Lectures on (abelian) Chern-Simons vortices

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Abstract

Various aspects including the construction and the symmetries of Abelian Chern-Simons vortices are reviewed. Extended version of the Lectures delivered at NIKHEF (Amsterdam), July 2006.

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Contents

1 Introduction : the Chern-Simons form.

The interaction between physics and mathematics can go in both ways. For example, Maxwell's theory, introduced to describe electromagnetism, has later been applied also to mathematics, namely to potential theory. This happened again when Maxwell's electromagnetism was generalized to describe non-Abelian interactions – and Yang-Mills theory became, later, an essential tool in differential geometry for studying the characteristic (Pontryagin) classes over even-dimensional manifolds.

With Chern-Simons theory history went the opposite way : in the early 1970, S. S. Chern and Simons [1] introduced their secondary characteristic classes to study bundles over odd-dimensional manifolds; this geometrical tool found subsequent application in low-dimensional physics. In 3 space-time dimensions (the only case we study in this Review), the (Abelian) Chern-Simons three-form is[1]

$$(\text{CS form}) = \frac{1}{4}A_\alpha F_{\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma \tag{1.1}$$

where $A = A_\alpha dx^\alpha$ is some vector potential.

The first applications of the Chern-Simons form to physics came in the early 1980, namely in *topologically massive gauge theory* [2, 3]. It has been realized that (1.1) can indeed be added to the usual Maxwell term in the electromagnetic action

$$S = S_{em} + S_{CS} = \int \left(\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} - \frac{\kappa}{4}\epsilon^{\alpha\beta\gamma}A_\alpha F_{\beta\gamma}\right) d^3x. \tag{1.2}$$

A novel feature is that while the 3-form (1.1) is *not* invariant under a gauge transformation $A_\alpha \to A_\alpha + \partial_\alpha\lambda$,

$$(\text{CS form}) \to (\text{CS form}) - \frac{\kappa}{4}\epsilon^{\alpha\beta\gamma}(\partial_\alpha\lambda)F_{\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma,$$

the field equations associated with S_{CS},

$$\partial_\alpha F^{\alpha\gamma} + \frac{\kappa}{2}\epsilon^{\alpha\beta\gamma}F_{\alpha\beta} = 0, \tag{1.3}$$

are gauge invariant. This is understood by noting that, using the sourceless Maxwell equation $\epsilon^{\alpha\beta\gamma}\partial_\alpha F_{\beta\gamma} = 0$, the Lagrangian action (1.2) is seen to change by a mere surface term,

$$\Delta L_{CS} = -\partial_\alpha\left(\frac{\kappa}{4}\epsilon^{\alpha\beta\gamma}F_{\beta\gamma}\lambda\right)$$

and defines, therefore, a fully satisfactory gauge theory in 3 dimensions. Moreover, it can be inferred from (1.3) that the Chern-Simons dynamics endows the gauge field A_μ with a "topological mass" [2].

[1] Three-dimensional space-time indices are denoted by $\alpha, \beta, \ldots$ The non-Abelian generalization (not considered here) of the Chern-Simons form is

$$\frac{1}{4}\mathrm{Tr}\left(A_\alpha F_{\beta\gamma} - \frac{2}{3}A_\alpha A_\beta A_\gamma\right) dx^\alpha \wedge dx^\beta \wedge dx^\gamma.$$

[2] The Chern-Simons form behaves in a way analogous to what happens for a Dirac monopole, for which no global vector potential exists, but the classical action is, nevertheless, satisfactorily defined. In the non-Abelian context and over a compact space-time manifold, this leads, in a way analogous to the Dirac quantization of the monopole charge, to the quantization of the Chern-Simons coefficient interpreted as the topological mass, [3, 4, 5].

The Chern-Simons term can be used, hence, to supplement the usual Maxwellian dynamics; it can even replace it altogether. The resulting dynamics is "poorer", since it allows no propagating modes. It has, in turn, larger symmetries : while the Maxwell term $(1/4)F_{\alpha\beta}F^{\alpha\beta}$ requires giving a metric $g_{\alpha\beta}$, the Chern-Simons term is *topological* : the integral

$$\int \frac{1}{4}\epsilon^{\alpha\beta\gamma}A_\alpha F_{\beta\gamma}d^3x$$

is independent of the coordinates we choose. Thus, while the Maxwell theory has historically been at the very origin of (special) relativity, the Chern-Simons term can allow both relativistic and non-relativistic (or even mixed) theories.

The large invariance of the Chern-Simons term lead, in the mid-eighties, to consider a Galilean field theory [6].

The main physical application of Chern-Simons gauge theory is, however, in condensed matter physics, namely to the *Quantum Hall Effect* [7, 8]. The latter, discovered in the early eighties [9], says that in a thin semiconductor in a perpendicular magnetic field the longitudinal resistance drops to zero if the magnetic field takes some specific values, called "plateaus". The current, $\vec{\jmath}$ and the electric field, $\vec{e}$, should satisfy in turn an "off-diagonal" relation of the form

$$\vec{\jmath} = \begin{pmatrix} 0 & -\kappa \\ \kappa & 0 \end{pmatrix} \vec{e}. \tag{1.4}$$

where the real coefficient κ, identified as the *Hall conductivity*, is, furthermore, *quantized*. In the integer Quantum Hall Effect (IQHE) κ is an integer multiple of some basic unit, while in the Fractional Quantum Hall Effect (FQHE), it is a rational multiple.

The explanation of this surprising and unexpected quantization, provided by Laughlin's "microscopic" theory, involves quasiparticles and quasiholes. (For details the reader is referred to the literature [9]). These are composite objects that carry both (fractional) electric charge and a magnetic flux : they are *charged vortices*.

The similarities of the Fractional Quantum Hall Effect (FQHE) with superfluidity lead condensed matter physicsist, however, to ask for a phenomenological effective theory of the FQHE [7]. Remember that the phenomenological description of 'ordinary' superconductivity and superfluidity is provided by Landau-Ginzburg theory [11] : the Cooper pairs formed by the electrons are represented by a scalar field, whose charge is twice that of the electron. The scalar fields interact through their electromagnetic fields, governed by the Maxwell equations. The theory admits static, finite-energy, vortex-like solutions [12].

Landau-Ginzburg theory does not involve the time as it "lives" in space alone. Its relativistic extension, called the Abelian Higgs model [13], admits again static and purely magnetic vortex-type solutions [14]. Note that ordinary Landau-Ginzburg does not admit any interesting non-relativistic extension, owing to the intrinsically relativistic character of the Maxwell dynamics.

Any "Landau-Ginzburg" theory of the FQHE must reproduce Hall's law (1.4). Now, as first pointed out in [10], adding the usual current term $j^\alpha A_\alpha$ to the action and suppressing the conventional Maxwell term, the spatial component of field equations become precisely *Hall law*, (1.4). This observation does not seem to have influenced condensed matter physicsist, though, who went in their own way to arrive, independently, at similar conclusions.

The evolution of Chern-Simons gauge theories has been parallel and (almost) unrelated in high-energy/mathematical physics and in condensed matter physics for at least a decade. It is interesting to compare the early progress in both fields : similar ideas arose, independently and

almost simultaneously, see Table 1. The main difference has been that while condensed matter physicist were more interested in the physical derivation and in its application to the Hall effect, high-energy/mathematical physicists explored the existence and the construction of solutions.

FIELD THEORY (hep-th)	CONDENSED MATTER (cond-mat)
1981 Schonfeld; Deser, Jackiw, Templeton topologically massive gauge theory	1980-1982 v. Klitzing et al.; Tsui, Stormer, Gossard Integer/Fractional Quantum Hall effect
1984-85 Hagen Galilei-invariant field theory in 2+1d; Jackiw, Friedman et al. relation to Hall effect	1983 Laughlin microscopic theory of FQHE ground-state wave functions
1986 Paul-Khare; De Vega-Schaposnik vortices in Maxwell/YM + CS	1986-87 Girvin-MacDonald effective 'Landau-Ginzburg' theory
1990-91 Hong et al, Jackiw et al. relativistic/non-relativistic topological/non-topological self-dual vortices	1989 Zhang, Hansson, Kivelson time-dependent LG theory with vortex solutions
1991 Ezawa et al., Jackiw-Pi vortices in external field	1993 Tafelmayer topological vortices in the Zhang model
1997 Manton NR Maxwell-CS	

Table 1: The Chern-Simons form in field theory and in condensed matter physics.

The first, static, 'Landau-Ginzburg' theory for the QHE has been put forward by Girvin [7] on phenomenological grounds. An improved and extended to time-dependent theory was derived from Laughlin's microscopic theory by Zhang, Hansson and Kivelson [8], see Section 2.

These theories involve, inevitably, the Chern-Simons form. In contrast to ordinary Landau-Ginzburg theory, they can accomodate relativistic as well as non-relativistic field theory is a strong argument in its favor : while high-energy theories are typically relativistic, condensed matter physics is intrinsically non-relativistic.

Below, we review various aspects of Chern-Simons gauge theory.

In detail, we first recall the way that lead condensed matter physicists to Chern-Simons theory, remarkably similar to that advocated by Feynman in his "Another point of view" presented in his 1962 Lectures on Statistical Mechanics [15].

Interrupting the condensed-matter-physics approach, the field theoretical aspects start with Section 3, devoted to relativistic topological vortices.

Their non-relativistic limit is physically relevant, owing to the intrinsically non-relativistic character of condensed matter physics. It also provides an explicitly solvable model. For a particular choice of the potential and for a specific value of the coupling constant, the second-order field equations can be solved by solving instead first order "self-duality" equations. The problem can in fact be reduced to solving the *Liouville equation.* Not all solutions are physically admissible, though : those which are correspond to *rational functions.* This provides as with a *quantization theorem of the magnetic charge*, as well as with a *parameter counting.*

The symmetry problem enters the theory at (at least) two occasions. Firstly, do the self-dual equations come from a Bogomolny-type decomposition of the energy ? This question becomes meaningful if a conserved energy-momentum tensor is constructed. Such a proceedure is canonical in a relativistic field theory, but is rather subtle in the non-relativistic context.

Another important application is to the following. Do we have other than non-self-dual solutions at the specific "self-dual" value of the coupling constant ? The (negative) answer is

obtained in a single line, if the conformal symmetry of the theory is exploited [42].

Similar ideas work for vortices in a constant background field, see Sec. 5. These models are important, since they correspond to those proposed in the Landau-Ginzburg theory of the Fractional Quantum Hall Effect [8].

Finally, we consider spinorial models. Again, explicit solutions are found and their symmetries are studied using the same techniques as above.

2 Landau-Ginzburg theory for the QHE

In Ref. [7], Girvin and MacDonald call, on phenomenological grounds, for a “Landau–Ginzburg” theory for the Quantum Hall Effect. On phenomenological grounds, they suggest to represent the off-diagonal long range order (ODLRO) by a scalar field $\psi(\vec{x})$ on the plane, and the frustration due to deviations away from the quantized Laughlin density by an effective gauge potential $\vec{a}(\vec{x})$. They propose to describe this static planar system by the Lagrange density

$$Ł_{GMD} = -\left|\vec{D}\psi\right|^2 + \phi\big(|\psi|^2 - 1\big) - \frac{\kappa}{2}\Big(\phi\, b + \vec{a}\times\vec{\nabla}\phi\Big), \tag{2.1}$$

where $b = \vec{\nabla}\times\vec{a}$ is the effective magnetic field, $\vec{D} = \vec{\nabla} - i\,\vec{a}$ is the gauge-covariant derivative, and the Lagrange multiplier ϕ is a scalar potential. The associated equations of motion read

$$\vec{D}^2\psi = \phi\,\psi, \tag{2.2}$$

$$\kappa\, b = |\psi|^2 - 1, \tag{2.3}$$

$$\kappa\,\vec{\nabla}\times\phi = \vec{\jmath}, \tag{2.4}$$

where

$$\vec{\jmath} = -i\big(\psi^*\vec{D}\psi - \psi(\vec{D}\psi)^*\big) \tag{2.5}$$

is the current.

The first of these equations is a static, gauged Schrödinger equation for the matter field.

The second is the relation proposed by Girvin and MacDonald to relate the magnetic field to the particle density.

The last equation is the Ampère–Hall law : $\vec{e} = -i\vec{\nabla}\phi$ is an effective electric field, so that (2.4) is indeed the Hall law (1.4), with κ identified as the *Hall conductance.*

Soon after, Zhang *et al.* [8] argued that the Girvin - MacDonalds model is merely a first step in the right direction and proposed a “better” Landau-Ginzburg model for the QHE, they derive directly form the microscopic theory [8, 16]. Their starting point is the Hamiltonian of a planar system of polarized electrons,

$$H_{pe} = \frac{1}{2m}\sum_a \Big[\vec{p}_a - e\vec{A}^{ext}(\vec{x}_a)\Big]^2 + \sum_a eA_0^{ext}(\vec{x}_a) + \sum_{a<b} V(\vec{x}_a - \vec{x}_b), \tag{2.6}$$

where A_α^{ext} is a vector potential for the constant external magnetic field B^{ext}, $A_i^{ext} = \frac{1}{2}B^{ext}\epsilon_{ij}x^j$ in the symmetric gauge. A_0^{ext} is the scalar potential for the external electric field, $E_i^{ext} = -\partial_i A_0$. V is the two-body interaction potential between the electrons. The common assumption is that V is Coulombian. The many-body wave function satisfies the Schrödinger equation

$$H_{pe}\Psi(\vec{x}_1,\ldots,\vec{x}_N) = E\,\Psi(\vec{x}_1,\ldots,\vec{x}_N) \tag{2.7}$$

and is assumed, by the Pauli principle, to be *totally antisymmetric* w.r.t. the interchange of any two electrons.

The clue of Zhang et al. [8] is to map the problem onto a *bosonic* one. Let us in fact consider the bosonic system with Hamiltonian

$$H_{bos} = \frac{1}{2m}\sum_a \Big[\vec{p}_a - e(\vec{A}^{ext}(\vec{x}_a) - \vec{a}(\vec{x}_a))\Big]^2 + \sum_a e\Big((A_0^{ext}(\vec{x}_a) + a_0(\vec{x}_a)\Big) + \sum_{a<b} V(\vec{x}_a - \vec{x}_b), \quad (2.8)$$

where the new vector potential, a_α, describes a gauge interaction of specific form among the particles,

$$\vec{a}(\vec{x}_a) = \frac{\Phi_0}{2\pi}\frac{\theta}{\pi}\sum_{b\neq a}\vec{\nabla}\gamma_{ab}, \quad (2.9)$$

where θ is a (for the moment unspecified) real parameter, and $\gamma_{ab} = \gamma_a - \gamma_b$ is the difference of the polar angles of electrons $\underline{a}$ and $\underline{b}$ w.r.t. some origin and polar axis. $\Phi_0 = h/ec$ is the flux quantum. The N-body bosonic wave function ϕ is required to be *symmetric* and satisfies

$$H_{bos}\phi(\vec{x}_1, \dots, \vec{x}_N) = E\,\phi(\vec{x}_1, \dots, \vec{x}_N). \quad (2.10)$$

Let us now consider the singular gauge transformation

$$\widetilde{\phi}(\vec{x}_1, \dots, \vec{x}_N) = U\,\phi(\vec{x}_1, \dots, \vec{x}_N), \qquad U = \exp\left[-i\sum_{a<b}\frac{\theta}{\pi}\gamma_{ab}\right]. \quad (2.11)$$

It is easy to check that

$$U\Big[\vec{p}_a - e(\vec{A}^{ext} - \vec{a})\Big]U^{-1} = \vec{p}_a - e\vec{A}^{ext} \implies U\,H_{bos}U^{-1} = H_{pe}, \quad (2.12)$$

so that ϕ satisfies (2.10) precisely when $\widetilde{\phi}$ satisfies the polarized-electron eigenvalue problem (2.7) with the same eigenvalue.

To conclude our proof, let us observe that $\widetilde{\phi}$ is antisymmetric precisely when the parameter θ is an *odd multiple* of π,

$$\theta = (2k+1)\pi. \quad (2.13)$$

Having replaced the fermionic problem by a bosonic one with the strange interaction (2.9), Zhang et al. proceed to derive a mean-field theory. Their model also involves a scalar field ψ coupled to both an external electromagnetic field A_μ^{ext} and to a "statistical" gauge field A_μ. It also includes a potential term, and is time-dependent. Their Lagrangian reads

$$\begin{aligned} \text{Ł}_{ZHK} &= -\tfrac{1}{4\theta}\epsilon^{\mu\nu\sigma}A_\mu\partial_\nu A_\sigma \\ &\quad +\psi^*\big[i\partial_t - (A_t + A_t^{ext})\big]\psi + \psi^*\big[-i\vec{\nabla} - (\vec{A} + \vec{A}^{ext})\big]^2\psi + U(\psi), \end{aligned} \quad (2.14)$$

where A_μ^{ext} is the vector potential of an external electromagnetic field and

$$U(\psi) = \mu|\psi|^2 - \lambda|\psi|^4 \quad (2.15)$$

is a self-interaction potential. The term $\mu|\psi|^2$ ($\mu \geq 0$) here is a chemical potential, while the quartic term is an effective interaction coming from the non-local expression

$$\tfrac{1}{2}\int \psi^\star(\vec{x})\psi^\star(\vec{x}')V(\vec{x}-\vec{x}')\psi(\vec{x})\psi(\vec{x}')d^2\vec{x}d^2\vec{x}'$$

in the second-quantized Hamiltonian, when the two-body potential is approximated by a delta function,

$$V(\vec{x}-\vec{x}')=-2\lambda\,\delta(\vec{x}-\vec{x}').$$

For a static system in a purely magnetic background and for $U(\psi)\equiv 0$, the two models are mathematically equivalent, though [57].

Let us point out that the ZHK Lagrangian is *first-order in the time derivative* of the scalar field. It is indeed non-relativistic, as will be shown in Section 5.

Zhang et al argue that their model admits vortex-type solutions [8, 16], studied in [19] in some detail.

3 Relativistic Chern-Simons vortices

Instead of pursuing the evolution in condensed matter physics, now we turn to study the field-theoretical aspects.

The first (Abelian)[3] Chern-Simons model is obtained by simply adding the Chern-Simons term to the usual Abelian Higgs model [20] :

$$L = \tfrac{1}{2}(D_\alpha\psi)^* D^\alpha\psi - U(\psi) - \tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + \tfrac{\kappa}{4}e^{\alpha\beta\gamma}A_\alpha F_{\beta\gamma}, \tag{3.1}$$

$$U(\psi) = \tfrac{\lambda}{2}\big(1-|\psi|^2\big)^2, \tag{3.2}$$

where $D_\alpha\psi = \partial_\alpha\psi - ieA_\alpha\psi$ is the covariant derivative, e the electric charge of the field ψ. and The Chern-Simons term is coupled through the coupling constant κ. The theory lives in $(2+1)$-dimensional Minkowski space, with the metric $(g_{\mu\nu}) = \mathrm{diag}\,(1,-1,-1)$, the coordinates being $x^0=t$ and $(x_i)=\vec{x}$.

The system can be studied along the same lines as in the Nielsen-Olesen case [13]. Paul and Khare [20] argue in fact that for the generalization to $A_0\neq 0$ of the Nielsen-Olesen radial Ansatz

$$A_0 = A_0(r), \qquad A_r = 0, \qquad A_\vartheta = -\frac{A(r)}{r}, \qquad \psi(r) = f(r)e^{-in\vartheta}, \tag{3.3}$$

the equations of motion,

$$\begin{aligned}
&\frac{d^2A}{dr^2} - \frac{1}{r}\frac{dA}{dr} - ef^2(n+eA) = \kappa r\frac{dA_0}{dr},\\
&\frac{d^2A_0}{dr^2} + \frac{1}{r}\frac{dA_0}{dr} - e^2A_0f^2 = \kappa\,\frac{1}{r}\frac{dA}{dr},\\
&\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{1}{r^2}\,(n+eA)^2\,f + e^2A_0^2f = -4\lambda f(1-f^2),
\end{aligned} \tag{3.4}$$

supplemented with the finite-energy asymptotic conditions

$$\begin{array}{llllllll}
\lim_{r\to\infty} f(r) & = & 1, & \lim_{r\to\infty} A(r) & = & -\dfrac{n}{e}, & \lim_{r\to\infty} A_0(r) & = \;\; 0\\
\lim_{r\to 0} f(r) & = & 0, & \lim_{r\to 0} A(r) & = & 0, & \lim_{r\to 0} A(r) & = \;\; 0
\end{array} \tag{3.5}$$

will admit a solution for each integer n. By (3.5) these solutions represent *charged topological vortices* sitting at the origin, since they carry both *quantized magnetic flux and electric charges*,

$$\Phi = \frac{2\pi}{e}\,n, \qquad Q = \kappa\,\frac{2\pi}{e}\,n = \kappa\,\Phi. \tag{3.6}$$

[3] A non-Abelian theory with vortex solutions has also been proposed, cf. [64, 27].

respectively.

While these vortices have interesting physical properties, the model suffers from the mathematical difficulty of having to solve second-order field equations. Further insight can be gained if we turn off the Maxwell term altogether, and trading the the standard fourth-order self-interaction scalar potential (3.2) for a 6^{th} order one,

$$U(\psi) = \frac{\lambda}{4}|\psi|^2\big(|\psi|^2-1\big)^2. \tag{3.7}$$

The Euler-Lagrange equations read

$$\tfrac{1}{2}D_\mu D^\mu \psi \quad = \quad -\frac{\delta U}{\delta \psi^*} \equiv -\frac{\lambda}{4}(|\psi|^2-1)\big(3|\psi|^2-1\big)\psi, \tag{3.8}$$

$$\tfrac{1}{2}\kappa\,\epsilon^{\mu\alpha\beta}F_{\alpha\beta} \quad = \quad ej^\mu, \tag{3.9}$$

where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the 'electromagnetic' field, and $j^\mu \equiv (\varrho, \vec{\jmath})$ is the current

$$j^\mu = \frac{1}{2i}\big[\psi^* D^\mu\psi - \psi(D^\mu\psi)^*\big]. \tag{3.10}$$

The first of the equations (3.8) is a nonlinear Klein-Gordon equation (NLKG), familiar from the Abelian Higgs model [13]; the second, (3.9), called the *Field-Curent Identity* (FCI), replaces the Maxwell equations. Let us observe that, unlike the latter, these equations are of the first order in the vector potentital.

It follows from the Bianchi identity, that the current (3.10) is conserved,

$$\epsilon^{\alpha\beta\gamma}\,\partial_\alpha F_{\beta\gamma} = 0, \quad \Rightarrow \quad \partial_\mu j^\mu = 0. \tag{3.11}$$

3.1 Finite-energy configurations

Let us consider a static field configuration (A_μ, ψ). The energy, defined as the space integral of the time-time component of the energy-momentum tensor associated with the Lagrangian, is

$$E \equiv \int d^2\vec{x}\, T^{00} = \int d^2\vec{x}\Big[\tfrac{1}{2}D_i\psi(D^i\psi)^* - \tfrac{1}{2}e^2A_0^2|\psi|^2 + \kappa A_0 B + U(\psi)\Big], \tag{3.12}$$

where $B = -F^{12}$ is the magnetic field. Note that this expression is *not* positive definite. Observe, however, that the static solutions of the equations of motion (3.8)-(3.9) are stationary points of the energy.

Variation of (3.12) w. r. t. A_0 yields one of the equations of motion, namely

$$-e^2A_0|\psi|^2 + \kappa B = 0. \tag{3.13}$$

Eliminating A_0 from (3.12) using this equation, we obtain the *positive definite* energy functional

$$E = \int d^2\vec{r}\,\Big[\tfrac{1}{2}D_i\psi(D^i\psi)^* + \frac{\kappa^2}{2e^2}\,\frac{B^2}{|\psi|^2} + U(\psi)\Big]. \tag{3.14}$$

We are interested in static, finite-energy configurations. Finite energy at infinity is guaranteed by the conditions[4]

$$\left\{\begin{array}{lllll} i.) & |\psi|^2-1 & = & \mathrm{o}\big(1/r\big), & \\ ii.) & B & = & \mathrm{o}\big(1/r\big), & r\to\infty. \\ iii.) & \vec{D}\psi & = & \mathrm{o}\big(1/r\big). & \end{array}\right. \tag{3.15}$$

[4]These conditions are in no way necessary; they yield the so-called *topological solitons*. Non-topological solutions are constructed in Ref. [24].

Therefore, the $U(1)$ gauge symmetry is broken for large r. In particular, the scalar field ψ is covariantly constant, $\vec{D}\psi = 0$. This equation is solved by parallel transport,

$$\psi(\vec{x}) = \exp\left[i\int_{\vec{x}_0}^{\vec{x}} eA_i dx^i\right]\psi_0, \tag{3.16}$$

which is well-defined whenever

$$\oint eA_i dx^i = \int_{\mathbf{R}^2} d^2\vec{x}\, eB \equiv e\Phi = 2\pi n, \qquad n = 0, \pm 1, \ldots. \tag{3.17}$$

Thus, *the magnetic flux is quantized.*

By i.), the asymptotic values of the Higgs field provide us with a mapping from the circle at infinity into the vacuum manifold, which is again a circle, $|\psi|^2 = 1$. Since the vector potential behaves asymptotically as

$$A_j \simeq -\frac{i}{e}\,\partial_j \log\psi, \tag{3.18}$$

the integer n in Eq. (3.17) is the *winding number* of this mapping; it is also called the *topological charge* (or *vortex number*).

Spontaneous symmetry breaking generates mass [25]. Expanding j^μ around the vacuum expectation value of ψ we find $j^\mu = -eA^\mu$ so that (3.9) is approximately

$$\tfrac{1}{2}\kappa\,\epsilon^{\mu\alpha\beta}F_{\alpha\beta} \approx -e^2 A^\mu. \quad \text{Hence} \quad F_{\alpha\mu} \approx -(e^2/\kappa)\,\epsilon_{\alpha\mu\beta}A^\beta.$$

Inserting here $F_{\alpha\beta}$ and deriving by ∂^α, we find that the gauge field A^μ satisfies the Klein-Gordon equation

$$\Box A^\mu \approx -\left(\frac{e^2}{\kappa}\right)^2 A^\mu,$$

showing that the mass of the gauge field is

$$m_A = \frac{e^2}{\kappa}. \tag{3.19}$$

The Higgs mass is found in turn expanding ψ around its expectation value, chosen as $\psi_0 = (1,0)$, $(\psi_r, \psi_\vartheta) = (1+\varphi, \theta)$, yielding

$$U = \underbrace{U(1)}_{=0} \quad + \quad \underbrace{\frac{\delta U}{\delta|\psi|}\Big|_{|\psi|=1}}_{=0}\varphi \quad + \quad \frac{1}{2}\underbrace{\left(\frac{\delta^2 U}{\delta|\psi|^2}\right)\Big|_{|\psi|=1}}_{m_\psi^2}\varphi^2, \tag{3.20}$$

since $|\psi| = 1$ is a critical point of U. We conclude that the mass of the Higgs particle is [5].

$$m_\psi^2 = \frac{\delta^2 U}{\delta|\psi|^2}\Big|_{|\psi|=1} = 2\lambda. \tag{3.21}$$

[5]This can also be seen by considering the radial equation (3.23) below.

3.2 Radially symmetric solutions

For the radially symmetric Ansatz

$$A_0 = A_0(r), \qquad A_r = 0, \qquad A_\vartheta = A(r), \qquad \psi(r) = f(r)e^{-in\vartheta}, \tag{3.22}$$

the equations of motion (3.8)-(3.9) read

$$\begin{aligned}
&\frac{1}{r}\frac{dA}{dr} + \frac{e^2}{\kappa}f^2 A_0 = 0,\\
&r\frac{dA_0}{dr} + \frac{e^2}{\kappa}f^2\left(\frac{n}{e} + A\right) = 0,\\
&\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{e^2}{r^2}\left(\frac{n}{e} + A\right)^2 f + e^2 A_0^2 f = -\frac{\lambda}{4}f(1-f^2)(1-3f^2),
\end{aligned} \tag{3.23}$$

with asymptotic conditions

$$\begin{array}{lllllll}
\lim_{r\to\infty} A(r) & = & -\dfrac{n}{e}, & \qquad & \lim_{r\to\infty} f(r) & = & 1,\\
\lim_{r\to 0} A(r) & = & 0, & \qquad & \lim_{r\to 0} f(r) & = & 0.
\end{array} \tag{3.24}$$

This is either seen by a direct substitution into the equations, or by re-writing the energy as

$$E = 2\pi\int_0^\infty dr\left\{\frac{r}{2}(f')^2 + \frac{a^2}{2r}f^2 + \frac{\kappa^2}{2e^4 f^2}\frac{(a')^2}{r} + U(f)\right\}, \tag{3.25}$$

where $a = eA + n$. The upper equation in (3.23) is plainly the radial form of (3.13). Then variation of (3.25) with respect to a and f yields the two other equations in (3.23).

Approximate solutions can be obtained by inserting the asymptotic value, $f \approx 1$, into the first two equations:

$$\frac{a'}{r} + \frac{e^2}{\kappa}A_0 = 0, \qquad A_0' + \frac{e^2}{\kappa}\frac{a}{r} = 0, \tag{3.26}$$

from which we infer that

$$\frac{d^2 A_0}{d\rho^2} + \frac{1}{\rho}\frac{dA_0}{d\rho} - A_0 = 0, \qquad \rho \equiv (e^2/\kappa)r. \tag{3.27}$$

This is the modified Bessel equation [Bessel equation of imaginary argument] of order zero. Hence

$$A_0 = CK_0\left(\frac{e^2}{\kappa}r\right). \tag{3.28}$$

Similarly, for $a = n/e + A$ we find, putting $\alpha = a/r$,

$$\alpha'' + \frac{\alpha'}{\rho} - \left(1 + \frac{1}{\rho^2}\right)\alpha = 0, \tag{3.29}$$

which is Bessel's equation of order 1 with imaginary argument. Thus $\alpha = CK_1(\rho)$ so that

$$A = -\frac{n}{e} + C\frac{e^2}{\kappa}r\,K_1\left(\frac{e^2}{\kappa}r\right). \tag{3.30}$$

Another way of deriving this result is to express A from the middle equation in (3.23),

$$A = -\frac{n}{e} - \frac{\kappa}{e^2}\,r\frac{d}{dr}A_0. \tag{3.31}$$

The consistency with (3.30) follows from the recursion relation $K_0' = -K_1$ of the Bessel functions.

An even coarser approximation is obtained by eliminating the $\frac{a'}{r}$ term by setting $a = ur^{-1/2}$ and dropping the terms with inverse powers of r. Then both equations reduce to

$$u'' = \big(\frac{e^2}{\kappa}\big)^2 u \qquad \Longrightarrow \qquad A_0 = a = \frac{C}{\sqrt{r}}\, e^{-m_A r}, \tag{3.32}$$

which shows that the fields approach their asymptotic values exponentially, with characteristic length determined by the gauge field mass.

The deviation of f from its asymptotic value, $\varphi = 1 - f$, is found by inserting φ into the last eqn. of (3.23); developping to first order in φ we get

$$\varphi'' + \frac{1}{r}\varphi' - 2\lambda\varphi \simeq 0 \qquad \Longrightarrow \qquad \varphi = CK_0(\sqrt{2\lambda}\, r), \tag{3.33}$$

whose asymptotic behaviour is again exponential with characteristic length $(m_\psi)^{-1}$,

$$\varphi = \frac{C}{\sqrt{r}}\, e^{-m_\psi r}. \tag{3.34}$$

The penetration depths of the gauge and scalar fields are therefore

$$\eta = \frac{1}{m_A} = \frac{e^2}{\kappa} \qquad \text{and} \qquad \xi = \frac{1}{m_\psi} = \frac{1}{\sqrt{2\lambda}}, \tag{3.35}$$

respectively. For small r instead, inserting the developments in powers of r, we find

$$\begin{aligned} f(r) &\sim f_0 r^{|n|} + \dots, \\ A_0 &\sim \alpha_0 - \frac{e f_0^2 n}{2\kappa|n|}\, r^{2|n|} + \dots, \\ A &\sim -\frac{e^2 f_0^2 \alpha_0}{2\kappa(|n|+1)}\, r^{2|n|+2} + \dots, \end{aligned} \tag{3.36}$$

where α_0 and f_0 are constants. In summary,

$$\begin{aligned} |\psi(r)| &\equiv f(r) &\propto& \begin{cases} r^{|n|} & r \sim 0 \\ 1 - Cr^{-1/2}\, e^{-m_\psi r} & r \to \infty \end{cases} \\ |E(r)| &= |A_0'(r)| &\propto& \begin{cases} r^{2|n|-1} & r \sim 0 \\ Cr^{-1/2}\, e^{-m_A r} + \text{lower order terms} & r \to \infty \end{cases} \\ |B(r)| &= \frac{|A'|}{r} &\propto& \begin{cases} r^{2|n|} & r \sim 0 \\ Cr^{-3/2}\, e^{-m_A r} + \text{lower order terms} & r \to \infty \end{cases} \end{aligned} \tag{3.37}$$

3.3 Self-dual vortices

In the Abelian Higgs model, an important step has been to recognize that, for a specific value of the coupling constant, the field equations could be reduced to first-order [21, 14]. This can also be achieved by a suitable modification of the model [23, 24], we discuss below in some detail.

Let us suppose that the fields have equal masses, $m_\psi^2 = m_A^2 \equiv m^2$ and hence equal penetration depths. Then the Bogomolny trick applies, i.e., the energy can be rewritten in the form

$$E = \int d^2\vec{r}\left[\tfrac{1}{2}|(D_1 \pm iD_2)\psi|^2 + \tfrac{1}{2}\Big|\frac{\kappa}{e}\frac{B}{\psi} \mp \frac{e^2}{2\kappa}\psi^*(1-|\psi|^2)\Big|^2\right] \mp \int d^2\vec{r}\,\frac{eB}{2}(1-|\psi|^2). \qquad (3.38)$$

The last term can also be presented as

$$\mp\frac{eB}{2} \mp \tfrac{1}{2}\vec{\nabla}\times\vec{\jmath}.$$

The integrand of the B-term yields the magnetic flux; the second is transformed, by Stokes' theorem, into the circulation of the current at infinity which vanishes, since all fields drop off at infinity by assumption. Its integral is therefore proportional to the magnetic flux, $\pm e\Phi/2$. Since the first integral is non-negative, we have, in conclusion,

$$E \geq \frac{e|\Phi|}{2} = \pi|n|, \qquad (3.39)$$

equality being only attained if the *self-duality equations*

$$D_1\psi = \mp iD_2\psi \qquad (3.40)$$

$$eB = \pm\frac{m^2}{2}\,|\psi|^2(1-|\psi|^2) \qquad (3.41)$$

hold. It is readily verified that the solutions of equations (3.13) and (3.40-3.41) solve automatically the non-linear Klein-Gordon equation (4.61).

Let us first study the radial case. For the Ansatz (3.22) the self-duality equations become

$$f' = \pm\frac{a}{r}f, \qquad \frac{a'}{r} = \pm\tfrac{1}{2}m^2 f^2(f^2-1), \qquad (3.42)$$

where we introduced again $a = eA + n$. Deriving the first of these equations and using the second one, for f we get the *Liouville - type equation*

$$\triangle \log f = \frac{m^2}{2}\,f^2(f^2-1). \qquad (3.43)$$

Another way of obtaining the first-order eqns. (3.42) is to rewrite, for

$$U(f) = \frac{m^2}{8}\,f^2\big(f^2-1\big)^2, \qquad (3.44)$$

the energy as

$$\pi\int_0^\infty rdr\left\{\big[f' \mp \frac{a}{r}f\big]^2 + \frac{1}{m^2f^2}\big[a'r \mp \frac{m^2}{2}f^2(f^2-1)\big]^2\right\} \pm \pi(af^2)\Big|_0^\infty \mp \pi a\Big|_0^\infty. \qquad (3.45)$$

The boundary conditions read

$$\begin{aligned} a(\infty) &= 0, \quad f(\infty) &= 1, \\ a(0) &= n, \quad f(0) &= 0, \end{aligned} \qquad (3.46)$$

and thus $E \geq \pi|n|$ as before, with equality attained iff the equations (3.42) hold.

For $n = 0$ the only solution is the vacuum,

$$f \equiv 1, \qquad A \equiv 1. \tag{3.47}$$

To see this, note that the boundary conditions at infinity are $f(\infty) = 1$ and $A(\infty) = 0$. Let now $f(r)$, $A(r)$ denote an arbitrary finite-energy configuration and consider

$$f_\tau(r) = f(r), \qquad A_\tau(r) = \tau A(r)$$

where $\tau > 0$ is a real parameter. This provides us with a 1-parameter family configurations with finite energy

$$E_\tau = 2\pi \int_0^\infty dr \left\{ \frac{r}{2} \left(f'\right)^2 + \tau^2 \left[\frac{a^2}{2r} f^2 + \frac{r}{2m^2} \left(\frac{a'}{rf}\right)^2 \right] + U(f) \right\},$$

which is a monotonic function of τ, whose minimum is at $\tau = 0$ i.e. for $a \equiv 0$. Then Eq. (3.42) implies that $f' \equiv 0$ so that $f \equiv 1$ is the only possibility.

Let us assume henceforth that $n \neq 0$. No analytic solution has been found so far. To study the large-r behaviour, put $\varphi \equiv 1 - f$. Inserting $f \approx 1$, Eqs. (3.42) reduce to

$$\varphi' = \mp \frac{a}{r}, \qquad \frac{a'}{r} = \mp m^2 \varphi.$$

Deriving, we get

$$\begin{aligned} \varphi'' + \tfrac{1}{r}\varphi' - m^2\varphi = 0 \quad &\Longrightarrow \quad \varphi = C K_0(mr), \\ a'' - \tfrac{1}{r}a' - m^2 a = 0 \quad &\Longrightarrow \quad a = C mr K_1(mr). \end{aligned}$$

Thus, for large r,

$$f \approx 1 - C K_0(mr) A \approx -\frac{n}{e} + C mr K_1(mr) \tag{3.48}$$

with some constant C. For small r instead, Eq. (3.42), yields, to $\mathrm{O}(r^{5|n|+1})$, the expansion

$$\begin{aligned} f(r) &= f_0 r^{|n|} - \frac{f_0^3 m^2}{2(2n+2)^2}\, r^{3|n|+2} + \mathrm{O}(r^{5|n|+2}). \\ A &= -\frac{f_0^2 m^2}{2(2|n|+2)e}\, r^{2|n|+2} + \frac{f_0^2 m^2}{2(4|n|+2)e}\, r^{4n+2} + \mathrm{O}(r^{4|n|+4}). \end{aligned} \tag{3.49}$$

The result is consistent with (3.36) since the constant α_0 is now $\alpha_0 = m/2e = e/2\kappa$.

Let us mention that the asymptotic behaviour expressed in Eq. (3.46) is actually valid in full generality, without the assumption of radial symmetry. Expressing in fact the vector-potential from the self-duality condition $(D_1 \pm iD_2)\psi = 0$ as

$$e\vec{A} = \vec{\nabla}(\mathrm{Arg}\,\psi) \pm \vec{\nabla} \times \log|\psi| \tag{3.50}$$

and inserting into the second equation in (3.42), we get again (3.46), with $|\psi|$ replacing f.

Index-theoretical calculations show that, for topological charge n, Eqn. (3.40-3.41) admits a $2|n|$ parameter family of solutions [23].

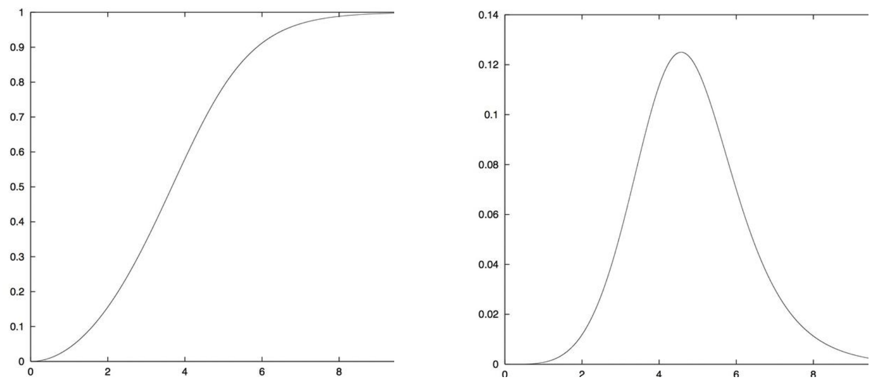

Figure 1: *The scalar and the magnetic fields of the radially symmetric charge-2 relativistic vortex. Note that $B = 0$ where the scalar field vanishes, so that the magnetic field has a doughnut-like shape.*

4 Non-relativistic vortices

The non-relativistic limit of the system studied in Section 3 is found [26, 27] by setting

$$\psi = e^{-imc^2t}\Psi + e^{+imc^2t}\bar{\Psi}, \tag{4.1}$$

where Ψ and $\bar{\Psi}$ denote the particles and antiparticles, respectively. Inserting (4.1) into the action, dropping the oscillating terms and only keeping those of order $1/c$, shows that both the particles and antiparticles are separately conserved. We can therefore consistently set $\bar{\Psi} = 0$. The remaining matter Lagrangian reads

$$\text{Ł}_{matter} = i\Psi^\star D_t\Psi - \frac{|\vec{D}\Psi|^2}{2m} + \frac{\Lambda}{2}(\Psi^\star\Psi)^2, \tag{4.2}$$

where $\Lambda = e^2/mc|\kappa|$. At first, we will let the constant Λ be arbitrary.

It will be shown in Section 4.3 below that the theory is non-relativistic see.

In what follows, we put $c = 1$.

Variation of $\displaystyle\int \text{Ł}_{matter}$ w. r. t. $\Psi^\star$ yields the *gauged non-linear Schrödinger equation*

$$i\partial_t\Psi = \left[-\frac{\vec{D}^2}{2m} - eA_t - \Lambda\Psi^\star\Psi\right]\Psi, \tag{4.3}$$

where $\vec{D} = \vec{\nabla} - ie\vec{A}$. A self-consistent system is obtained by adding the matter action to the Chern-Simons action (1.1). The variational equations are the Chern-Simons equations (3.9) written in non-relativistic notations,

$$B \qquad \equiv \epsilon^{ij}\partial_i A^j = -\frac{e}{\kappa}\,\varrho, \qquad \text{Gauss} \tag{4.4}$$

$$E^i \quad \equiv -\partial_i A^0 - \partial_t A^i = \frac{e}{\kappa}\epsilon^{ij}J^j, \qquad \text{FCI} \tag{4.5}$$

where

$$\varrho = \Psi^*\Psi \quad \text{and} \quad J^\mu \equiv (\varrho, \vec{J}) = (\Psi^\star\Psi, \frac{1}{2mi}\big[\Psi^*\vec{D}\Psi - \Psi(\vec{D}\Psi)^*\big]) \tag{4.6}$$

are the density and the current, respectively. The invariance of (4.2) w.r.t. global gauge transformations $\Psi \to e^{i\omega}\Psi$ implies the continuity equation

$$\partial_t \varrho + \vec{\nabla} \cdot \vec{J} = 0. \tag{4.7}$$

Eqns. (4.4) and (4.5) are called the Gauss' law and the field-current identity (FCI), respectively.

4.1 Self-dual NR vortex solutions

We would again like to find static soliton solutions. The construction of an energy-momentum tensor is now more subtle, because the theory is non-relativistic. A conserved energy-momentum tensor can, nevertheless, be constructed [29, 26, 34, 35], see Section 4.3 below. It provides us with the energy functional

$$E = \int d^2x \left(\frac{|\vec{D}\Psi|^2}{2m} - \frac{\Lambda}{2}(\Psi^\star\Psi)^2 \right). \tag{4.8}$$

Now we apply once again the Bogomolny trick. Using the identity

$$|\vec{D}\Psi|^2 = |(D_1 \pm iD_2)\Psi|^2 \pm m\vec{\nabla} \times \vec{J} \pm eB\,\varrho, \tag{4.9}$$

the energy (4.8) is written as

$$E = \int d^2x \, \frac{\big|(D_1 \pm iD_2)\Psi|^2}{2m} - \tfrac{1}{2}\left(\Lambda - \frac{e^2}{m|\kappa|}\right)(\Psi^\star\Psi)^2. \tag{4.10}$$

The energy is, hence, positive definite if

$$\Lambda \leq \frac{e^2}{m|\kappa|}, \tag{4.11}$$

that we assume henceforth. The vacuum is clearly

$$\vec{A} = 0, \qquad \Psi = 0. \tag{4.12}$$

To get finite energy, the following large-r asymptotic behaviour is required :

$$\begin{cases} \vec{D}\Psi \to 0 \\ \\ |\Psi| \to 0 \end{cases} \qquad \text{as} \quad r \to \infty. \tag{4.13}$$

The second condition here implies that the finite-energy vortices constructed below are *non-topological* : $\Psi|_\infty : S_\infty \to 0$.

For the specific value [6]

$$\Lambda = \frac{e^2}{m|\kappa|} \tag{4.14}$$

of the non-linearity, in particular, the second term vanishes. Then the absolute minimum of the energy, namely zero, is attained for *self-dual* or *antiself-dual fields* [7], i.e. for such that

$$D_\pm\Psi = 0, \qquad \text{where} \qquad D_\pm = D_1 \pm iD_2. \tag{4.15}$$

[6] (4.14) is the same as the one we obtained above by taking the non-relativistic limit of the self-dual relativistic theory.

[7] The equations (4.15-4.19) can indeed be derived, by symmetry reduction, from the $4D$ self-dual Yang-Mills equations [36].

Do we get a static solution of the problem by minimizing the energy ? Let us first observe that the energy functional (4.8) does not include the time component, A_t, which should be fixed by the field equations. For the self-dual Ansatz (4.15), the current is expressed as

$$\vec{J} = \pm\frac{1}{2me}\vec{\nabla}\times\varrho. \tag{4.16}$$

Using another identity, namely

$$\vec{D}^2\Psi = \left(D_+D_- + eB\right)\Psi, \tag{4.17}$$

the static field equations can be written as

$$\begin{aligned}\left[\frac{D_+D_-}{2m} + \left(\Lambda \mp \frac{e^2}{2m\kappa}\right)\varrho - eA_t\right]\Psi = 0 \qquad (4.18)\\ \kappa B - e\varrho = 0 \qquad (4.19)\\ \vec{\nabla}\left(A_t - \frac{1}{2m\kappa}\varrho\right) = 0 \qquad (4.20)\end{aligned}$$

By inspection, using $\big[D_+, D_-\big] = eB$, we infer that a static solution is obtained, for the specific value (4.14), for

$$\begin{aligned}D_\pm\Psi = 0 \qquad (4.21)\\ \kappa B + e\Psi\Psi^* = 0, \qquad (4.22)\end{aligned}$$

when the time component of the potential is

$$A_t = \frac{1}{2m\kappa}\varrho. \tag{4.23}$$

Separating the phase as $\Psi = e^{ie\omega}\sqrt{\varrho}$, the SD equation is solved by

$$\vec{A} = \frac{1}{2e}\vec{\nabla}\times\log\varrho + \vec{\nabla}\omega. \tag{4.24}$$

When we insert $B = \vec{\nabla}\times\vec{A}$ into (4.22) ϱ has to solve the *Liouville equation*,

$$\boxed{\triangle\log\varrho = \pm\frac{2e^2}{\kappa}\varrho,} \tag{4.25}$$

cf. (3.43).

Having solved this equation, the scalar and vector potentials are given by (4.23) and (4.24), respectively. In the latter, the phase ω has to be chosen so that it cancels the singularity due to the zeros of ϱ. This property is related to the quantization of the vortex charge, see Section 4.2 below. The vectorpotential will be given in (4.44) below.

4.2 Vortex solutions of the Liouville equation

The vortices are hence constructed out of the solutions of the Liouville equation (4.25). Solutions defined over the whole plane arise when the r.h.s. is negative. Hence, the upper sign has to be chosen when $\kappa < 0$ and the lower sign when $\kappa > 0$. Then the general solution reads

$$\boxed{\varrho = \frac{4|\kappa|}{e^2}\frac{|f'|^2}{(1+|f|^2)^2},} \tag{4.26}$$

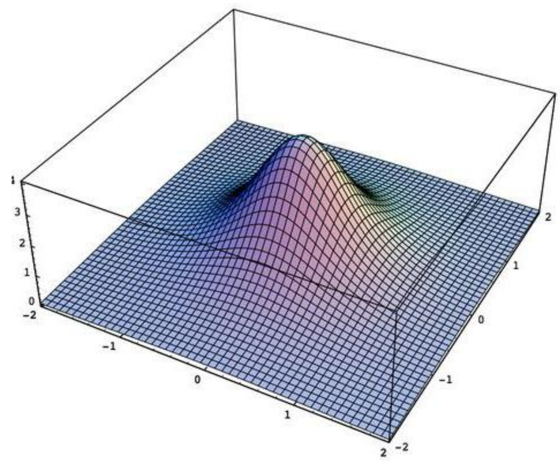

Figure 2: *The non-relativistic radially symmetric $N = 1$ vortex has a maximum at $r = 0$.*

where f is a meromorphic function of $z = x + iy$. For the radial Ansatz

$$f(z) = z^{-N} \tag{4.27}$$

we obtain, in particular, the radially symmetric solution

$$\varrho(r) = \frac{4N^2|\kappa|}{e^2}\,|\frac{r^{-2(N+1)}}{(1+r^{-2N})^2}. \tag{4.28}$$

The regularity requires, furthermore, that N be an integer at least 1. For $N = 1$ (Fig. 4.2), the origin is a maximum of ϱ; for $N \geq 2$, it is a zero : the vortex has a "doughnut-like" shape, see Fig. 4.2. Presenting (4.28) as

$$\varrho(r) = \frac{4N^2|\kappa|}{e^2}\left[\frac{r^{(N-1)}}{1+r^{2N}}\right]^2 \tag{4.29}$$

shows, forthermore, that the density ϱ [and hence the magnetic field B] vanishes at the origin, $\varrho(0) = 0 = B(0)$, except for $N = 1$, cf. the figures. Owing to the Gauss law (4.4), the magnetic field behaves as in fact as

$$B \propto -\varrho \sim r^{2(N-1)}. \tag{4.30}$$

The singularity in the first term in the vector potential A (4.24) can be canceled choosing the phase of ψ as

$$\omega = (N-1)\theta, \tag{4.31}$$

where θ is the polar angle of the position vector [29].

Returning to the general case, we observe that not all meromorphic function yield a physically interesting solution, though. The natural requirement is that the magnetic and scalar fields, B and Ψ, must be regular, and that the magnetic flux,

$$\Phi = \int B\, d^2x, \tag{4.32}$$

be finite. For the radial Ansatz (4.27) we find, for example,

$$\Phi = -\frac{4\pi\, N(\mathrm{sg}\,\kappa)\hbar}{e}. \tag{4.33}$$

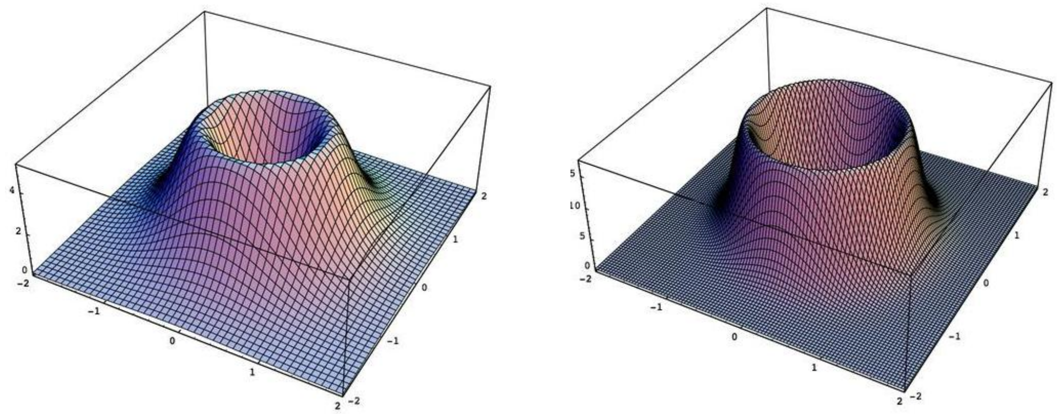

Figure 3: *For $N \geq 2$, the non-relativistic radially symmetric a vortices have a 'doughnut-like' shape: the particle density vanishes at $r = 0$. The figure shows those with $N = 2$ and $N = 4$.*

Which functions f yield regular, finite-flux solutions ? How can we calculate the flux ? Is it quantized ? How many independent solutions do we get for a fixed value Φ ? The answers in [38, 39] are not entirely satisfactory: on the one hand, the proof given in [38] is based on an asymptotic behaviour, that is only valid in the radial case. On the other hand, the parameter-counting given in [39] uses an index theorem, which is an unnecessary complication here, when explicit solutions are known. Elementary proofs were found in [40].

Theorem 1 [40] : *The meromorphic function $f(z)$ yields a regular vortex solution with finite magnetic flux if and only if $f(z)$ is a rational function,*

$$f(z) = \frac{P(z)}{Q(z)} \qquad \text{s.t.} \qquad \deg P < \deg Q, \tag{4.34}$$

where the highest-order term on Q can be normalized to 1.

In particular, when all roots of $Q(z)$ are simple, $f(z)$ can be developed into partial fractions,

$$f(z) = \sum_{i=1}^{N} \frac{c_i}{z - z_i}, \tag{4.35}$$

where the c_i and the z_i are $2n$ complex numbers, we get the $4N$-parameter family of N separated one-vortices [29]. Note that this formula breaks down for superimposed vortices.

The proof proceeds through a series of Lemmas [40], and amounts to showing that f can only have a finite number of isolated singularities that can not be essential neither at a finite point, nor at infinity. Then a theorem of complex analysis [45] says that f is necessarily rational.

The density (4.26) is readily seen to be invariant w.r.t.

$$f \to \frac{f + c}{1 - \bar{c} f}. \tag{4.36}$$

In particular, taking c imaginary and letting it go to infinity, it is invariant under changing f into $1/f$. Hence $\deg P \leq \deg Q$ can be assumed. But $\deg P = \deg Q$ can be eliminated by a suitable redefinition [29].

Theorem 2 [40] : *The magnetic flux of the solution generated by P/Q is evenly quantized,*

$$\Phi = 2N(\operatorname{sign}\kappa)\Phi_0, \qquad N = \deg Q, \qquad \Phi_0 = -2\pi\frac{\hbar}{e}. \tag{4.37}$$

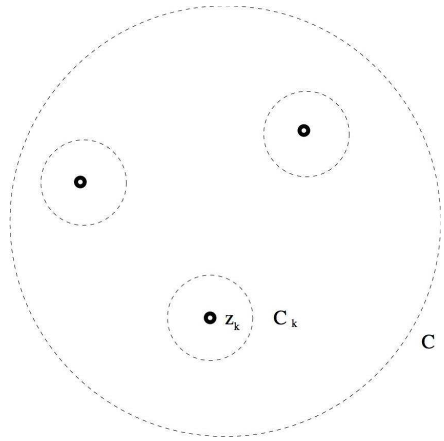

Figure 4: *The charge of a vortex is proportional to the sum of the multiplicities the zeros of the denominator $Q(z)$ in (4.34).*

The proof amounts to showing that only the roots of the denominator

$$Q(z) = (z - z_1)^{n_1} \dots (z - z_m)^{n_m}, \qquad (\sum_k n_k = N) \tag{4.38}$$

contribute to the charge. (4.37) is inferred by transforming the flux (4.32) into a contour integral along the circle at infinity C. The isolated zeros of $Q(z)$, $z_1, \dots, z_m$, are identified with the "positions" of the vortices. Each of them can be encircled by disjoint circles C_k, and the charge comes form these zeros,

$$\Phi = \oint_C = \sum_k \oint_{C_k} = \sum_k n_k \left(-(\mathrm{sg}\,\kappa) \frac{4\pi\hbar}{e} \right) = -2N(\mathrm{sg}\,\kappa)\Phi_0. \tag{4.39}$$

Let us fix $N = \deg Q > \deg P$.

<u>Theorem 3.</u> : *The solution generated by (4.34) depends on* $4N - 1$ *parameters, where* N *is the degree of the denominator* $Q(z)$.

The proof follows at once from Theorem 1. : N is the degree of the denominator, $Q(z)$, which, being normalized, has N complex coefficients. Due to deg $P <$ deg Q, the numerator also has N complex coefficients (some of which can vanish). The (-1), (missed in [40]) comes from noting that, by (4.26), the general phase of f is irrelevant, so that the highest coefficient of P can be chosen to be real.

These results have a rather elegant geometric interpretation [41]. A rational function

$$w = \frac{P(z)}{Q(z)} = \frac{a_m z^m + \dots + a_0}{b_n z^n + \dots + b_0} \tag{4.40}$$

$(a_m, b_n \neq 0)$ always has a limit as $z \to \infty$, namely ∞ if $m = \deg Q > \deg Q = n$, a_m/b_n if $m = n$, and zero, if $m < n$. It extends therefore as a mapping, still denoted by f, between the Riemann spheres,

$$f \ : S_z \to S_w, \tag{4.41}$$

obtained by compactifying the complex z and w-planes by adding the point at infinity. Then the z and w are stereographic coordinates. The w-sphere carries, in particular, the canonical

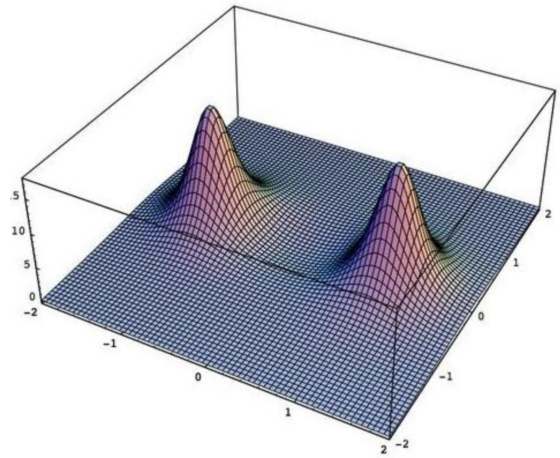

Figure 5: *Two separated* 1*-vortices with charge* $N = 2$*.*

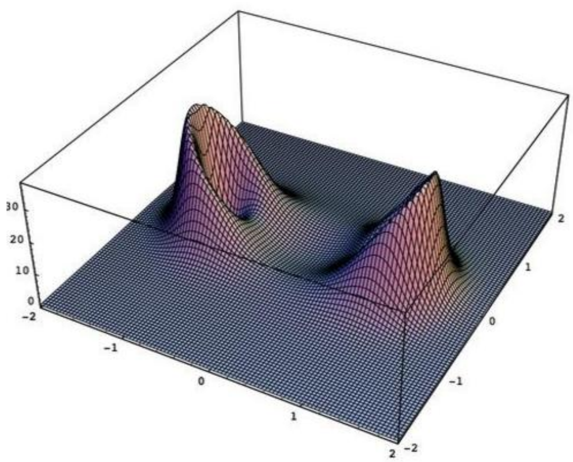

Figure 6: *Two separated charge-2 vortices with total charge* $N = 4$*.*

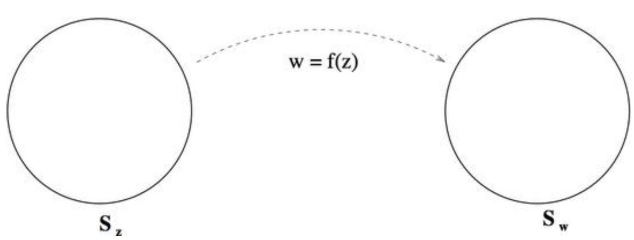

Figure 7: *The magnetic charge of a non-topological vortex is in fact the topological charge in monopole theory.*

surface form

$$\Omega = 2i\frac{dw\wedge d\bar{w}}{(1+w\bar{w})^2}. \tag{4.42}$$

Using the Gauss law $B = -(\frac{e}{\kappa})\varrho$, the magnetic flux of the vortex, $\Phi = \displaystyle\int Bd^2x$, is [8]

$$\Phi = -(\mathrm{sg}\,\kappa/e)\int\frac{4|f'|^2}{(1+|f|^2)^2}d^2z = -(2/e)(\mathrm{sg}\,\kappa)\int_{S_z} f^*\Omega, \tag{4.43}$$

where we recognize the *topological charge* of monopole theory [46]. The integral in (4.43) is in fact the same as the homotopy class of the mapping $f:\ S_z \to S_w$.

Equivalently, the magnetic charge is the *Brouwer degree* of f [which is the number of times the image is covered].

Generalizing (4.31), the regularity of the vector potential requires chosing the phase ω so that [40]

$$(\partial_x - i\partial_y)\omega = \sum_{i=1}^{N_Q}\frac{n_i-1}{z-z_i} + \sum_{k=1}^{N_P}\frac{n_k+1}{z-Z_k}. \tag{4.44}$$

where the z_i, $i=1,\dots n_Q$ are the distinct roots of the denominator $Q(z)$ and n_i their respective multiplicity, so that $\sum_{i=1}^{N_Q} n_i = \deg Q = N$ is the vortex number. The Z_k; $k=1,\dots N_P$, are the roots of the numerator; their multiplicities are m_k, and $\sum_{k=1}^{N_P} m_k = \deg P < N$.

Remarkably, the self-dual solutions of the Ø(3) non-linear sigma model ([47]) are, once again, precisely those described here.

4.3 Symmetries of non-relativistic vortices

A subtle point of non-relativistic CS theory is the construction of a conserved energy-momentum tensor. Jackiw and Pi [29] present the rather complicated-looking expressions

$$\begin{aligned}
T^{00} &= \tfrac{1}{2m}|\vec{D}\Psi|^2 - \tfrac{\Lambda}{2}|\Psi]^4,\\
T^{i0} &= -\tfrac{1}{2}\Big((\vec{D}_t\Psi)^*(D_i\Psi) + (D_i\Psi)^*D_t\Psi\Big),\\
T^{0i} &= -\tfrac{i}{2}\left(\Psi^*D_i\Psi - (D_i\Psi)^*\Psi\right),\\
T^{ij} &= -\tfrac{1}{2}\Big((D_i\Psi)^*D_j\Psi + (D_j\Psi)^*D_i\Psi - \delta_{ij}\,|\vec{D}\Psi|^2\Big)\\
&\quad +\tfrac{1}{4}\left(\delta_{ij}\,\triangle - 2\partial_i\partial_j\right)(|\Psi|^2) + \delta_{ij}\,T^{00},
\end{aligned} \tag{4.45}$$

whose conservation,

$$\partial_\alpha T^{\alpha\beta} = 0, \tag{4.46}$$

can be checked by a diract calculation. The tensor $T^{\alpha\beta}$ is, however, symmetric only in the spatial indices,

$$T^{0i} \neq T^{i0}, \qquad T^{ij} = T^{ji}. \tag{4.47}$$

T^{ij} has been "improved" and satisfies, instead of the usual tracelessness-condition $T^\alpha_{\ \alpha} = 0$ of relativistic field theory, the modified trace condition

$$T^i_{\ i} = 2T^{00}. \tag{4.48}$$

[8] Remember that in our units $2\pi\hbar = h = 1$.

These unusual properties are, as we explain it below, hallmarks of Schrödinger, rather then Lorentz-conformal invariance.

Let us remind the Reader the definition : a *symmetry* is a transformation which interchanges the solutions of the coupled equations of motion. For a Lagrangian system, an infinitesimal space-time symmetry can be represented by a vector field X^μ on space-time, is a symmetry, when it changes the Lagrangian by a surface term,

$$Ł \to Ł + \partial_\alpha K^\alpha \tag{4.49}$$

for some function K. To each such transformation, Nœther's theorem associates a conserved quantity, namely

$$C = \int \left(\frac{\delta Ł}{\delta(\partial_t \chi)} \delta\chi - K^t \right) d^2\vec{x}, \tag{4.50}$$

where χ denotes, collectively, all fields.

The Galilean symmetry of our Chern-Simons-theory follows from the general framework [6]. To each generator of the centrally-extended Galilei group is associated a conserved quantity, namely

$$\begin{array}{lll}
\mathcal{H} = \displaystyle\int T^{00} d^2x, & \quad & \text{energy} \\
\mathcal{P}_i = \displaystyle\int T^{0i}\, d^2x, & & \text{momentum} \\
\mathcal{J} = \displaystyle\int \epsilon_{ij} x^i T^{0j},\, d^2x & & \text{angular momentum} \\
\mathcal{G}_i = t\mathcal{P}_i - m\displaystyle\int x_i \varrho d^2x & & \text{center of mass} \\
\mathcal{N} = m\displaystyle\int x_i \varrho d^2x & & \text{mass (particle number)}
\end{array} \tag{4.51}$$

What is less expected is that the model admits two more conserved generators, namely

$$\begin{array}{lll}
\mathcal{D} = t\mathcal{H} - \frac{1}{2} x_i \mathcal{P}_i & \quad & \text{dilatation} \\
\mathcal{K} = -t^2\mathcal{H} + 2t\mathcal{D} + \dfrac{m}{2}\displaystyle\int r^2 \varrho\, d^2x & & \text{expansion}
\end{array} \tag{4.52}$$

The Poisson brackets,

$$\Big\{f, g\Big\} = \int \sum_i \left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial p_i} \right) d^2x \tag{4.53}$$

of these conserved quantities are those which define the non-relativistic "conformal" extension of the Galilei group, called the Schrödinger group [43]

$$\begin{array}{lllllllll}
\big\{\mathcal{G}_i, \mathcal{G}_j\big\} &=& 0, & \big\{\mathcal{P}_i, \mathcal{P}_j\big\} &=& 0, & \big\{\mathcal{P}_i, \mathcal{G}_j\big\} &=& \delta_{ij}\,\mathcal{N}, \\
\big\{\mathcal{G}_i, \mathcal{R}\big\} &=& \epsilon_{ij}\,\mathcal{G}_j, & \big\{\mathcal{P}_i, \mathcal{R}\big\} &=& \epsilon_{ij}\mathcal{P}_j, & & & \\
\big\{\mathcal{H}, \mathcal{G}_i\big\} &=& \mathcal{P}_i, & \big\{\mathcal{H}, \mathcal{P}_i\big\} &=& 0, & \big\{\mathcal{H}, \mathcal{R}\big\} &=& 0, \\
\big\{\mathcal{H}, \mathcal{D}\big\} &=& 2\mathcal{H}, & \big\{\mathcal{H}, \mathcal{K}\big\} &=& \mathcal{D}, & \big\{\mathcal{D}, \mathcal{K}\big\} &=& 2\mathcal{K}, \\
\big\{\mathcal{R}, \mathcal{D}\big\} &=& 0, & \big\{\mathcal{R}, \mathcal{K}\big\} &=& 0, & \big\{\mathcal{D}, \mathcal{G}_i\big\} &=& \mathcal{G}_i, \\
\big\{\mathcal{D}, \mathcal{P}_i\big\} &=& -\mathcal{P}_i, & \big\{\mathcal{K}, \mathcal{G}_i\big\} &=& 0, & \big\{\mathcal{K}, \mathcal{P}_i\big\} &=& \mathcal{G}_i.
\end{array} \tag{4.54}$$

In particular, $\mathcal{D}$ and $\mathcal{K}$ span, with the energy, $\mathcal{H}$, an $\emptyset(2,1)$ subgroup.

Additional symmetries play an important rôle [42]. Deriving the expansion generator $\mathcal{K}$ in (4.52) twice w.r.t. time shows in fact that

$$\left(\frac{m}{2}\int r^2|\Psi|^2 d^2x\right)''$$

is twice the Hamiltonian, and is hence time-independent. It follows that for fields that make $|\Psi|^2$ time-independent, in particular for static fields, the energy vanishes. Therefore, when Λ takes the specific "self-dual" value (4.14) the solution is necessarily self-dual by eqn (4.10) [9].

We mention for the record that applying any symmetry transformation to a solution of the field equations yields another solution. For example, a boost or an expansion applied to the static solution $\Psi_0(\vec{X})$ of Jackiw and Pi produces time-dependent solutions,

$$\Psi(T,\vec{X}) = \frac{1}{1-kT}\exp\left\{-\frac{i}{2}\Big[2\vec{X}\cdot\vec{b}+T\vec{b}^2+k\frac{(\vec{X}+\vec{b}T)^2}{1-kT}\Big]\right\}\Psi_0(\frac{\vec{X}+\vec{b}T}{1-kT}). \tag{4.55}$$

4.4 Symmetries in the non-relativistic Kaluza-Klein-type framework

How do the extra symmetries come about ? Can one derive the energy-momentum tensor (4.45), together with its strange properties (4.47), in a systematic way ? This is conveniently achieved in the "non-relativistic Kaluza-Klein" framework. The clue is that non-relativistic spacetime can be obtained from a $(3+1)$ dimensional relativistic spacetime, M, endowed with a Lorentz-signature metric $g_{\mu\nu}$ and a covariantly constant, lightlike vector ξ^μ. (Such a manifold, called a "Bargmann space", is in fact a gravitational pp wave [30]. These spaces can provide exact string vacua [32]). Then non-relativistic spacetime is the factor space of M, obtained by factoring out the integral curves of ξ^μ.

When M is the Minkowski space, in particular, the metric can be written using light-cone coordinates t and s as

$$ds^2 = d\vec{x}^2 + 2dtds. \tag{4.56}$$

More generally, we can have

$$ds^2 = g_{ij}dx^idx^j + 2dtd(s+A_idx^i) - 2Udt^2, \tag{4.57}$$

where g_{ij} is some spatial metric and A_i and U are a vector and a scalar potential, respectively.

The coordinate $\vec{x}$ can be viewed as position, t as non-relativistic time, and s as an "internal, Kaluza-Klein-type coordinate", directed along the "vertical" vector $\xi^\mu = \partial_s$. Quotienting M by the integral curves of ξ^μ amounts, intuitively, to "forgetting" s.

It is now easy to check that *the projection of the null-geodesics* of M, endowed with the metric

$$ds^2 = g_{ij}dx^idx^j + 2dtd(s+A_idx^i) - 2Udt^2, \tag{4.58}$$

satisfy the usual equations of motion of a non-relativistic particle in a (static) "electromagnetic" field

$$\vec{B} = \text{curl}\ \vec{A}, \qquad \vec{E} = -\text{grad}\ U. \tag{4.59}$$

With one strange detail, though : the coupling constant is not the electric charge, e, but the mass, m.

[9] In the Abelian Higgs model the analogous theorem is rather difficult to prove [14].

For $\vec{A}=0$, in particular, we recover, as noticed by Eisenhart in 1929 [31], Newton's equations.

Null geodesics are conformally invariant and their projections are hence invariant w.r.t. ξ-preserving conformal transformations which are, hence, symmetries of the projected system.

In Minkowski space (4.56), in particular, the (infinitesimal) conformal transformations span the conformal algebra $\o(4,2)$; those which preserve the lightlike vector $\xi^\mu=\partial_s$ are precisely the generators of the (centrally extended) *planar Schrödinger group*, centrally extended with the mass (the standard central extension of the Galilei group).

This "non-relativistic Kaluza-Klein" framework has been useful to study the Schrödinger symmetry of classical systems, and can also adapted to CS field theory [34]. Let us choose indeed, on M, a four-vector potential a_μ with field strength $f_{\mu\nu}$ and let j_μ be a four-current.

• Let us posit the relation

$$\kappa f_{\mu\nu}=e\sqrt{-g}\epsilon_{\mu\nu\rho\sigma}\xi^\rho j^\sigma. \tag{4.60}$$

Then $f_{\mu\nu}$ is the lift from space-time with coordinates $\vec{x}$ and t of a closed two-form $F_{\mu\nu}$. a_μ can be chosen therefore as the pull-back of a 3-potential $A_\alpha=(A_t,\vec{A})$. The four-current j^μ projects in turn onto a 3-current $J^\alpha=(\varrho,\vec{J})$. Then (4.60) is readily seen to project precisely to the Chern-Simons equations (4.4)-(4.5).

• Similarly, let ψ denote a scalar field on M and let us posit the (massless) non-linear Klein-Gordon wave equation

$$\left[D_\mu D^\mu-\frac{R}{6}+\lambda(\psi^*\psi)\right]\psi=0, \tag{4.61}$$

where $D_\mu=\nabla_\mu-iea_\mu$ is the metric and gauge covariant derivative on M and we have also added, for the sake of generality, a term which involves the scalar curvature, R of M. Requiring that the scalar field be equivariant,

$$\xi^\mu D_\mu\psi=im\psi, \tag{4.62}$$

$\Psi=e^{ims}\psi$ will be a function of $\vec{x}$ and t alone, and (4.61) becomes, for the Minkowski metric (4.56), the gauged non-linear Schrödinger equation (4.3).

• The systems (4.60) and (4.61) are coupled through

$$j^\mu=\frac{1}{2mi}\left[\psi^*(D^\mu\psi)-\psi(D^\mu\psi)^*\right], \tag{4.63}$$

that projects to the relation (4.6).

Eqns. (4.60)-(4.61)-(4.62)-(4.63) form a self-consistent system allowing us to lift our non-relativistic coupled scalar field-Chern-Simons system to the relativistic spacetime M. It can be now shown [34] that the latter is invariant w.r.t. any conformal transformation of the metric of M that preserves the "vertical" vector ξ^μ. Thus, we have just established the Schrödinger invariance of the non-relativistic Chern-Simons + scalar field system.

The theory on M is relativistic and admits, therefore, a conserved, traceless and symmetric energy-momentum tensor $\theta_{\mu\nu}$. In the present case, the canonical procedure yields

$$\begin{aligned}3m\theta_{\mu\nu} \;=\;& (D_\mu\psi)^*D_\nu\psi+D_\mu\psi(D_\nu\psi)^*\\ &-\tfrac{1}{2}\left(\psi^*D_\mu D_\nu\psi+\psi(D_\mu D_\nu)^*\right)\\ &+\tfrac{1}{2}|\psi|^2\left(R_{\mu\nu}-\tfrac{R}{6}g_{\mu\nu}\right)-\tfrac{1}{2}g_{\mu\nu}(D^\rho\psi)^*D_\rho\psi-\tfrac{\lambda}{4}g_{\mu\nu}|\psi|^4.\end{aligned} \tag{4.64}$$

It is now easy to prove that

$$T^{00} = -\theta^0{}_0, \quad T^{i0} = -\theta^i{}_0 - \tfrac{1}{6m}\partial_i\partial_t\varrho,$$
$$T^{0j} = \theta^0{}_j \qquad T^{ij} = \theta^i{}_j + \tfrac{1}{3m}\left(\delta^i_j\Delta - \partial^i\partial_j\right)\varrho, \tag{4.65}$$

where Δ is the spatial Laplace operator. These formulae allow us to infer all those properties of $T^{\alpha\beta}$ listed above.

In Ref. [34] a version of Noether's theorem was proved. It says that, for any ξ-preserving conformal vectorfield (X^μ) on Bargmann space, the quantity

$$Q_X = \int_{\Sigma_t} \vartheta_{\mu\nu} X^\mu \xi^\nu \sqrt{\gamma}\, d^2\vec{x}, \tag{4.66}$$

is a constant of the motion. (Here γ is the determinant of the metric g_{ij} induced by $g_{\mu\nu}$ on 'transverse space' $t = \text{const.}$.) The charge (4.66) is conveniently calculated using

$$\vartheta_{\mu\nu}\xi^\nu = \frac{1}{2i}\left[\psi^*\left(D_\mu\psi\right) - \psi\left(D_\mu\psi\right)^*\right] - \frac{1}{6}\,\xi_\mu\left(\frac{R}{6}|\psi|^2 + \left(D^\nu\psi\right)^* D_\nu\psi + \frac{\lambda}{2}|\psi|^4\right). \tag{4.67}$$

It is worth mentionning that chosing the "vertical" vector ξ^μ spacelike would provide us with a relativistic theory "downstairs".

It is interesting to note that our proof used the field equations. Is it possible to extend it to the variational principle ? On M we could use in fact the 4d "Chern-Simons type" expression

$$\frac{\kappa}{2}\epsilon^{\mu\nu\rho\sigma}\xi_\mu a_\nu f_{\rho\sigma}. \tag{4.68}$$

Curiously, this correctly reproduces the *relativistic* Chern-Simons equations (3.9) if ξ^μ is spacelike, but fails when it is lightlike, $\xi_\mu\xi^\mu = 0$ [33] — which is precisely the *non-relativistic* case we study here.

4.5 Time-dependent vortices in an external electromagnetic field

The static, non-relativistic Chern-Simons solitons studied above can be generalized to yield time-dependent vortex solutions in a constant external magnetic field $\mathcal{B}$ [58, 59]. Putting $\omega = \mathcal{B}/2$, the equation to be solved is [10]

$$i\big(D_\omega\big)_t\Psi_\omega = \left\{-\frac{1}{2}\vec{D}_\omega^2 - \Lambda\,\Psi_\omega^*\Psi_\omega\right\}\Psi_\omega. \tag{4.69}$$

Here the modified covariant derivative means

$$\big(D_\omega\big)_\alpha = \partial_\alpha - i(A_\omega)_\alpha - i\mathcal{A}_\alpha \tag{4.70}$$

$(\alpha = 0, 1, 2)$, where $\mathcal{A}_\alpha$ is a vector potential for the constant magnetic field, chosen to be

$$\mathcal{A}_0 = 0, \qquad \mathcal{A}_i = \tfrac{1}{2}\epsilon_{ij}x^j\mathcal{B} \equiv \omega\epsilon_{ij}x^j$$

$(i, j = 1, 2)$. $(A_\omega)_\alpha$ is the "statistical" vector potential of Chern-Simons electromagnetism, whose field strength is required to satisfy the field-current identities

$$B_\omega \equiv \epsilon^{ij}\partial_i A^j_\omega = -\frac{1}{\kappa}\varrho_\omega \tag{4.71}$$

$$E^i_\omega \equiv -\partial_i A^0_\omega - \partial_t A^i_\omega = \frac{1}{\kappa}\epsilon^{ij}J^j_\omega \tag{4.72}$$

[10] We use here units where $e = m = 1$.

with $\varrho_\omega = \Psi_\omega^*\Psi_\omega$ and $\vec{J}_\omega = (1/2i)[\Psi^*\vec{D}_\omega\Psi_\omega - \Psi_\omega(\vec{D}_\omega\Psi_\omega)^*]$.

These equations can be solved [58, 59] applying a coordinate transformation to a solution, Ψ and A_α, of the "free" problem with $\omega = 0$, according to [11]

$$\Psi_\omega(t,\vec{x}) \;=\; \frac{1}{\cos\omega t}\,\exp\left\{-i\omega\frac{r^2}{2}\tan\omega t\right\}\,\Psi(\vec{X},T), \tag{4.73}$$

$$(A_\omega)_\alpha \;=\; A_\beta\frac{\partial X^\beta}{\partial x^\alpha}, \tag{4.74}$$

with

$$T = \frac{\tan\omega t}{\omega}, \qquad \vec{X} = \frac{1}{\cos\omega t}R(\omega t)\,\vec{x}. \tag{4.75}$$

where $R(\theta)$ is the matrix of a planar rotation with angle θ.

A similar construction works in a harmonic background [59].

Now we explain the above results in our "Kaluza-Klein-type" framework introduced in the previous Section. Let us indeed consider coupled system (4.60)-(4.61)-(4.62)-(4.63) on a general "Bargmann" metric (4.58). Easy calculation shows that, after reduction, the covariant derivative

$$D_\alpha = \nabla_\alpha - iea_\alpha \tag{4.76}$$

(where ∇_α is the gauge-covariant derivative) becomes precisely $(D_\omega)_\alpha$ in (4.70), with, perhaps, a nontrivial A_0. The equation of motion is therefore an generalization of (4.69).

• Let us consider, for example, the " oscillator" metric

$$d\vec{x}_{osc}^2 + 2dt_{\rm osc}ds_{\rm osc} - \omega^2 r_{\rm osc}^2 dt_{\rm osc}^2, \tag{4.77}$$

where $\vec{x}_{\rm osc} \in \mathbf{R}^2$, $r_{\rm osc} = |\vec{x}_{\rm osc}|$ and ω is a constant. Its null geodesics correspond in fact to a non-relativistic, spinless particle in an oscillator background [30]. Requiring equivariance, (4.62), the wave equation (4.61) reduces to

$$i\partial_{t_{\rm osc}}\Psi_{\rm osc} = \left\{-\frac{\vec{D}^2}{2} + \frac{\omega^2}{2}{r_{\rm osc}}^2 - \Lambda\,\Psi_{\rm osc}\Psi_{\rm osc}^*\right\}\Psi_{\rm osc} \tag{4.78}$$

($\vec{D} = \vec{\partial} - i\vec{A}$, $\Lambda = \lambda/2$), which describes Chern-Simons vortices in a harmonic-force background, studied in Ref. [59].

• Let us consider instead the "magnetic" metric

$$d\vec{x}^2 + 2dt\Big[ds + \tfrac{1}{2}\epsilon_{ij}\mathcal{B}x^j dx^i\Big], \tag{4.79}$$

where $\vec{x} \in \mathbf{R}^2$ and $\mathcal{B}$ is a constant, whose null geodesics describe a charged particle in a uniform magnetic field in the plane [30]. Imposing equivariance, Eq. (4.61) reduces to Eq. (4.69) with $\Lambda = \lambda/2$ and the covariant derivative D_ω in Eq. (4.70).

Returning to the general theory, let φ denote a conformal Bargmann diffeomorphism between *two* Bargmann spaces, i.e. let $\varphi\ :\ (M,g,\xi) \to (M',g',\xi')$ be such that

$$\varphi^\star g' = \Omega^2 g \qquad \xi' = \varphi_\star\xi. \tag{4.80}$$

[11] The formulae in [28] also involve the factor $\exp\left\{i\frac{\mathcal{N}}{2\pi\kappa}\omega t\right\}$, as a result of gauge fixing.

Such a mapping projects to a diffeomorphism of the quotients, Q and Q' we denote by Φ. Then the same proof as in Ref. [34] allows one to show that if (a'_μ, ψ') is a solution of the field equations on M', then

$$a_\mu = (\varphi^\star a')_\mu \qquad \psi = \Omega\, \varphi^\star \psi' \tag{4.81}$$

is a solution of the analogous equations on M. Locally

$$\varphi(t, \vec{x}, s) = (t', \vec{x}', s') \quad \text{with} \quad (t', \vec{x}') = \Phi(t, \vec{x}), \quad s' = s + \Sigma(t, \vec{x}),$$

so that $\psi = \Omega\, \varphi^\star \psi'$ reduces to

$$\Psi(t, \vec{x}) = \Omega(t)\, e^{i\Sigma(t,\vec{x})}\, \Psi'(t', \vec{x}'), \qquad A_\alpha = \Phi^\star A'_\alpha \tag{4.82}$$

$(\alpha = 0, 1, 2)$. Note that φ takes a ξ-preserving conformal transformation of (M, g, ξ) into a ξ'-preserving conformal transformation of (M', g', ξ'). Conformally related Bargmann spaces have therefore isomorphic symmetry groups.

The conserved quantities can be related by comparing the expressions in (4.66). Using the transformation properties of the scalar curvature, short calculation shows that the conserved quantities associated to $X = (X^\mu)$ on (M, g, ξ) and to $X' = \varphi_\star X$ on (M', g', ξ') coincide,

$$Q_X = \varphi^\star Q'_{X'}. \tag{4.83}$$

The labels of the generators are, however, different (see the examples below).

- As a first application, we note the the lift to Bargmann space of Niederer's mapping [60]

$$\varphi(t_{\rm osc}, \vec{x}_{\rm osc}, s_{\rm osc}) = (T, \vec{X}, S),$$

$$T = \frac{\tan\omega\, t_{\rm osc}}{\omega}, \qquad \vec{X} = \frac{\vec{x}_{\rm osc}}{\cos\omega t_{\rm osc}}, \qquad S = s_{\rm osc} - \frac{\omega r_{\rm osc}^2}{2}\tan\omega t_{\rm osc} \tag{4.84}$$

carries the oscillator metric (4.77) Bargmann-conformally ($\varphi_\star \partial_{s_{\rm osc}} = \partial_S$) into the free form (4.56), with conformal factor $\Omega(t_{\rm osc}) = |\cos\omega t_{\rm osc}|^{-1}$. A solution in the harmonic background can be obtained by Eq. (4.81).

A subtlety arises, though: the mapping (4.84) is many-to-one : it maps each 'open strip'

$$I_j = \big\{(\vec{x}_{\rm osc}, t_{\rm osc}, s_{\rm osc}) \,\big|\, (j - \tfrac{1}{2})\pi < \omega t_{\rm osc} < (j + \tfrac{1}{2})\pi\big\}, \qquad j = 0, \pm1, \dots \tag{4.85}$$

corresponding to a *half oscillator-period*, onto full Minkowski space. Application of (4.81) with Ψ an 'empty-space' solution yields, in each I_j, a solution, $\Psi_{\rm osc}^{(j)}$. However, at the contact points $t_j \equiv (j + 1/2)(\pi/\omega)$, these fields may not match. For example, for the 'empty-space' solution obtained by an expansion, Eq. (4.55) with $\vec{b} = 0$, $k \neq 0$,

$$\lim_{t_{\rm osc}\to t_j - 0} \Psi_{\rm osc}^{(j)} = (-1)^{j+1}\frac{\omega}{k} e^{-i\frac{\omega^2}{2k} r_{\rm osc}^2}\,\Psi_0(-\frac{\omega}{k}\vec{x}) = -\lim_{t_{\rm osc}\to t_j + 0} \Psi_{\rm osc}^{(j+1)}. \tag{4.86}$$

The lef-and right limits differ hence by a sign. The continuity of the wave functions is restored including the 'Maslov' phase correction [62] :

$$\begin{aligned}
\Psi_{\rm osc}(t_{\rm osc}, \vec{x}_{\rm osc}) &= (-1)^j \frac{1}{\cos\omega t_{\rm osc}} \exp\left\{-\frac{i\omega}{2} r_{\rm osc}^2 \tan\omega t_{\rm osc}\right\} \Psi(T, \vec{X}) \\
(A_{\rm osc})_0(t_{\rm osc}, \vec{x}_{\rm osc}) &= \frac{1}{\cos^2\omega t_{\rm osc}} \big[A_0(T, \vec{X}) - \omega \sin\omega t_{\rm osc}\; \vec{x}_{\rm osc} \cdot \vec{A}(T, \vec{X})\big], \\
\vec{A}_{\rm osc}(t_{\rm osc}, \vec{x}_{\rm osc}) &= \frac{1}{\cos\omega t_{\rm osc}} \vec{A}(T, \vec{X}),
\end{aligned} \tag{4.87}$$

Eq. (4.87) extends the result in [59], which are only valid for $|t_{\rm osc}|<\pi/2\omega$, to any $t_{\rm osc}$ ([12]).

Since the oscillator metric (4.77) is Bargmann-conformally related to Minkowski space, Chern-Simons theory in the oscillator background has again a Schrödinger symmetry – but with "distorted" generators. The latter are in fact

$$J_{\rm osc}=\mathcal{J},\qquad H_{\rm osc}=\mathcal{H}+\omega^2\mathcal{K},\qquad N_{\rm osc}=\mathcal{N} \tag{4.88}$$

completed by

$$(C_{\rm osc})_\pm=\big(\mathcal{H}-\omega^2\mathcal{K}\pm 2i\omega\,\mathcal{D}\big)\,,\qquad (\vec{P}_{\rm osc})_\pm=\Big(\vec{\mathcal{P}}\pm i\omega\,\vec{\mathcal{G}}\Big)\,. \tag{4.89}$$

Let us observe in particular that the oscillator-Hamiltonian, $H_{\rm osc}$, is a combination of the "empty-space" !$\omega=0$) Hamiltonian and expansion, etc.

• Turning to the magnetic case, let us observe that the "magnetic" metric (4.79) is readily transformed into an oscillator metric (4.77), namely by the mapping $\varphi(t,\vec{x},s)=(t_{\rm osc},\vec{x}_{\rm osc},s_{\rm osc})$,

$$t_{\rm osc}=t,\qquad x^i_{\rm osc}=x^i\cos\omega t+\epsilon^i_jx^j\sin\omega t,\qquad s_{\rm osc}=s \tag{4.90}$$

[which amounts to switching to a rotating frame with angular velocity $\omega=\mathcal{B}/2$]. The vertical vectors $\partial_{s_{\rm osc}}$ and ∂_s are permuted.

Composing the two steps, we see that the time-dependent rotation (4.90), followed by the transformation (4.84), [which projects to the coordinate transformation (4.75)], carries conformally the constant-$\mathcal{B}$ metric (4.79) into the free ($\omega=0$)-metric. It carries therefore the 'empty' space solution $e^{is}\Psi$ with Ψ as in (4.55) into that in a uniform magnetic field background according to Eq. (4.81). Taking into account the equivariance, we get the formulæ of [58], multiplied with the Maslov factor $(-1)^j$.

Our framework also allows to 'export' the Schrödinger symmetry to non-relativistic Chern-Simons theory in the constant magnetic field background. The (rather complicated) generators [61] can be obtained using Eq. (4.83). For example, time-translation $t\to t+\tau$ in the $\mathcal{B}$-background amounts to a time translation for the oscillator with parameter τ plus a rotation with angle $\omega\tau$. Hence

$$H_{\mathcal{B}}=H_{\rm osc}-\omega\mathcal{J}=\mathcal{H}+\omega^2\mathcal{K}-\omega\mathcal{J}.$$

Similarly, a space translation for $\mathcal{B}$ amounts, in 'empty' space, to a space translations and a rotated boost : $P^i_{B}=\mathcal{P}^i+\omega\,\epsilon^{ij}\mathcal{G}^j$, etc.

All our preceding results apply to any Bargmann space which can be Bargmann-conformally mapped into Minkowski space. Now we describe all these 'Schrödinger-conformally flat' spaces. In $D=n+2>3$ dimensions, conformal flatness is guaranteed by the vanishing of the conformal Weyl tensor $C^{\mu\nu}{}_{\rho\sigma}$. Skipping technical details, we state that Schrödinger-conformal flatness requires [37]

$$\mathcal{A}_i=\tfrac{1}{2}\epsilon_{ij}\mathcal{B}(t)x^j+a_i,\qquad \vec{\nabla}\times\vec{a}=0,\qquad \partial_t\vec{a}=0, \tag{4.91}$$

$$U(t,\vec{x})=\tfrac{1}{2}C(t)r^2+\vec{F}(t)\cdot\vec{x}+K(t). \tag{4.92}$$

The metric (4.57)-(4.92) describes a uniform magnetic field $\mathcal{B}(t)$, an attractive $[C(t)=\omega^2(t)]$ or repulsive $[C(t)=-\omega^2(t)]$ isotropic oscillator and a uniform force field $\vec{F}(t)$ in the plane, all of

[12]For the static solution in [28] or for that obtained from it by a boost, $\lim_{t_{\rm osc}\to t_j}\Psi^{(j)}_{\rm osc}=0$, and the inclusion of the correction factor is not mandatory.

which may depend on time. It also includes a curlfree vector potential $\vec{a}(\vec{x})$ that can be gauged away if the transverse space is simply connected: $a_i = \partial_i f$ and the coordinate transformation $(t, \vec{x}, s) \to (t, \vec{x}, s + f)$ results in the 'gauge' transformation

$$\mathcal{A}_i \to \mathcal{A}_i - \partial_i f = -\tfrac{1}{2}\mathcal{B}\,\epsilon_{ij}x^j. \tag{4.93}$$

If, however, space is not simply connected, we can also include an external Aharonov-Bohm-type vector potential.

Being conformally related, all these metrics share the symmetries of flat Bargmann space: for example, if the transverse space is $\mathbf{R}^2$ we get the full Schrödinger symmetry; for $\mathbf{R}^2 \setminus \{0\}$ the symmetry is reduced rather to $\mathrm{o}(2) \times \mathrm{o}(2,1) \times \mathbf{R}$, as found for a magnetic vortex [44].

The case of a constant electric field is quite amusing. Its metric, $d\vec{x}^2 + 2dtds - 2\vec{F}\cdot\vec{x}dt^2$, can be brought to the free form by switching to an accelerated coordinate system,

$$\vec{X} = \vec{x} + \tfrac{1}{2}\vec{F}\,t^2, \quad T = t, \quad S = s - \vec{F}\cdot\vec{x}\,t - \tfrac{1}{6}\vec{F}^2t^3. \tag{4.94}$$

This example also shows that the action of the Schrödinger group — e.g. a rotation — looks quite differently in the inertial and in the moving frames.

In conclusion, our 'non-relativistic Kaluza-Klein' approach provides a unified view on the various known constructions and explains the common origin of their symmetries.

5 Non-relativistic Maxwell-Chern-Simons Vortices

Generalizing previous work [53, 54], Manton [56] proposed a modified version of the Landau-Ginzburg model for describing Type II superconductivity. His Lagrange density is a subtle mixture blended from the usual Landau-Ginzburg expression, augmented with the Chern-Simons term:

$$\begin{aligned}\mathcal{L} = \;& -\tfrac{1}{2}B^2 + \gamma\tfrac{i}{2}\big(\phi^* D_t\phi - \phi(D_t\phi)^*\big) - \tfrac{1}{2}\big|\vec{D}\phi\big|^2 - \tfrac{\lambda}{8}\big(1 - |\phi|^2\big)^2 \\ & +\mu\big(Ba_t + E_2a_1 - E_1a_2\big) - \gamma a_t - \vec{a}\cdot\vec{J}^T,\end{aligned} \tag{5.1}$$

where μ, $\gamma > 0$, $\lambda > 0$ are constants, $D_t\phi = \partial_t\phi - ia_t\phi$ and $D_i\phi = \partial_i\phi - ia_i\phi$ are the partial derivates, $B = \partial_1a_2 - \partial_2a_1$ is the magnetic field and $\vec{E} = \vec{\nabla}a_t - \partial_t\vec{a}$ is the electric field.

This Lagrangian has the usual symmetry-breaking quartic potential, but differs from the standard expression in that

1. it is linear in $D_t\phi$ but quadratic in $D_i\phi$;
2. the Maxwellian electric term $\vec{E}^2$ is missing;
3. includes the "weird" terms $-\gamma a_t$ and $-\vec{a}\cdot\vec{J}^T$, where $\vec{J}^T$ is the (constant) transport current.

Properties (1) and (2) stem from the requirement of Galilean rather than Lorentz invariance. The term $-\gamma a_t$ results in modifying the Gauss law (eqn. (5.4) below); the term $-\vec{a}\cdot\vec{J}^T$ is then needed in order to restore the Galilean invariance. To be so, the transport current has to transform as $\vec{J}^T \to \vec{J}^T + \gamma\vec{v}$ under a Galilei boost [56].

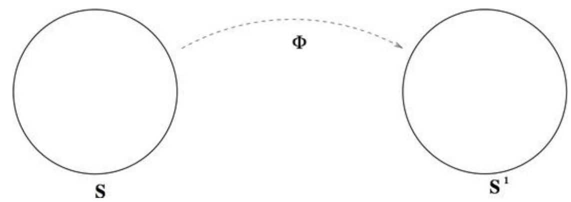

Figure 8: *The asymptotic values of the scalar field provide us with a mapping of the "circle at infinity" into the unit cercle $|\psi| = 1$. The winding number of the mapping is the topological charge, related to quantized magnetic flux.*

The field equations derived from (5.1) read

$$i\gamma D_t\phi = -\tfrac{1}{2}\vec{D}^2\phi - \tfrac{\lambda}{4}\big(1-|\phi|^2\big)\phi, \tag{5.2}$$

$$\epsilon_{ij}\partial_j B = J_i - J_i^T + 2\mu\,\epsilon_{ij}\,E_j, \tag{5.3}$$

$$2\mu B = \gamma\big(1-|\phi|^2\big), \tag{5.4}$$

where the (super)current is $\vec{J} = (1/2i)\big(\phi^*\vec{D}\phi - \phi(\vec{D}\phi)^*\big)$.

• The matter field satisfies hence a gauged, planar non-linear Schrödinger equation.

• The second equation is Ampère's law without the displacement current, as usual in the "magnetic-type" Galilean electricity [55].

• The last equation (which replaces the Gauss law of Maxwellian dynamics) is the (modified) "Field-Current Identity".

Manton [56] observed that when $\vec{J}^T = 0$, $\lambda = 1$ and $\mu = \gamma$, these same solutions yield magnetic vortices with $a_t = 0$, also in the Chern-Simons-modified model. Below, we generalize Manton's results to construct solutions with a non-vanishing electric field.

Before searching for solutions, let us discuss the finite-energy conditions. In the frame where $\vec{J}^T = 0$, the energy associated to the Lagrangian (5.1) is [57]

$$H = \int \Big\{\tfrac{1}{2}\big|\vec{D}\phi\big|^2 + \tfrac{1}{2}B^2 + U(\phi)\Big\}\, d^2\vec{x}, \qquad U(\phi) = \frac{\lambda}{8}\big(1-|\phi|^2\big)^2. \tag{5.5}$$

Eliminating the magnetic term $B^2/2$ using the Gauss law (5.4) results in a mere shift of the coefficient of the non-linear term,

$$H = \int \Big\{\tfrac{1}{2}\big|\vec{D}\phi\big|^2 + \frac{\Lambda}{8}\big(1-|\phi|^2\big)^2\Big\}\, d^2\vec{x}, \qquad \Lambda = \lambda + \frac{\gamma^2}{\mu^2}. \tag{5.6}$$

Finite energy "requires", just like in the Landau-Ginzburg case,

$$\vec{D}\phi \to 0 \qquad \text{and} \qquad |\phi|^2 \to 1, \tag{5.7}$$

By eqn. (5.7) we get, hence, *topological vortices* : the asymptotic values of scalar field provide us with a mapping from the circle at infinity S into the vacuum manifold $|\phi|^2 = 1$ which is again a circle,

$$\psi\Big|_\infty : S \to S^1. \tag{5.8}$$

The first of the equations in (5.7) implies that the angular component of vector potential behaves

asymptotically as n/r. The integer n here is also the winding number of the mapping defined by the asymptotic values of ϕ into the unit circle,

$$n = \frac{1}{2\pi}\oint_S \vec{a}\cdot d\vec{\ell} = \frac{1}{2\pi}\int B\, d^2\vec{x}. \tag{5.9}$$

The *magnetic flux is therefore quantized* and is related to the particle number

$$N \equiv \int \big(1-|\phi|^2\big)\, d^2\vec{x} = \frac{2\mu}{\gamma}\int B\, d^2\vec{x} = 4\pi\big(\frac{\mu}{\gamma}\big)\, n \tag{5.10}$$

by (5.4). N is conserved since the supercurrent satisfies the continuity equation $\partial_t\varrho + \vec{\nabla}\cdot\vec{J} = 0$.

5.1 Self-dual Maxwell-Chern-Simons vortices

Conventional Landau-Ginzburg theory admits finite-energy, static, purely magnetic vortex solutions. For a specific value of the coupling constant, one can find solutions by solving instead the first-order "Bogomolny" equations [12, 13, 21],

$$\begin{array}{rcl} (D_1 + iD_2)\phi &=& 0, \\ 2B &=& 1 - |\phi|^2. \end{array} \tag{5.11}$$

In the frame where $\vec{J}^T = 0$ (which can always be achieved by a Galilei boost), the static Manton equations read

$$\begin{aligned} &\gamma a_t \phi = -\tfrac{1}{2}\vec{D}^2\phi - \frac{\lambda}{4}\big(1-|\phi|^2\big)\phi, \\ &\vec{\nabla}\times B = \vec{J} + 2\mu\vec{\nabla}\times a_t, \\ &2\mu B = \gamma\big(1-|\phi|^2\big). \end{aligned} \tag{5.12}$$

Let us try to solve these equations by the first-order Ansatz

$$\begin{aligned} &(D_1 \pm iD_2)\phi = 0, \\ &2\mu B = \gamma\big(1-|\phi|^2\big). \end{aligned} \tag{5.13}$$

From the first of these relations we infer that

$$\vec{D}^2 = \mp i\big[D_1, D_2\big] = \mp B\vec{J} = \mp\tfrac{1}{2}\vec{\nabla}\times\varrho,$$

where $\varrho = |\phi|^2$. Inserting into the non-linear Schrödinger equation we find that it is identically satisfied when

$$a_t = (\pm 1/4\mu - \lambda/4\gamma)(1-\varrho).$$

Then from Ampère's law we get that λ has to be

$$\lambda = \pm 2\frac{\gamma}{\mu} - \frac{\gamma^2}{\mu^2}. \tag{5.14}$$

The scalar potential is thus

$$a_t = \tfrac{1}{4\mu}\big(\mp 1 + \tfrac{\gamma}{\mu}\big)\big(1-\varrho\big). \tag{5.15}$$

The vector potential is expressed using the "self-dual" (SD) Ansatz (5.13) as

$$\vec{a} = \pm\tfrac{1}{2}\vec{\nabla}\times\log\varrho + \vec{\nabla}\omega, \tag{5.16}$$

where ω is an arbitrary real function chosen so that $\vec{a}$ is regular. Inserting this into the Gauss law, we end up with the "Liouville-type" equation

$$\triangle \log \varrho = \pm\alpha\big(\varrho - 1\big), \qquad \alpha = \frac{\gamma}{\mu}.$$

Now, if we want a "confining" (stable) and bounded-from-below scalar potential, λ has to be positive. Then we see from eq. (5.14) that for the upper sign this means $0 < \alpha < 2$, whereas for the lower sign $-2 < \alpha < 0$. In any of the two cases (α positive or negative), the coefficient of $(\varrho - 1)$ in the r. h. s. is always positive: in the upper sign, it is α with $\alpha > 0$, in the lower sign, it is $-\alpha$ with $\alpha < 0$. We consider henceforth

$$\triangle \log \varrho = |\alpha|\big(\varrho - 1\big); \tag{5.17}$$

the magnetic and electric fields can be obtained from (5.16) and (5.15). Note that the electric field, $\vec{E} = \vec{\nabla} a_t$, only vanishes for $\mu = \pm\gamma$, i.e., when $\lambda = 1$, which is Manton's case.

The self-duality equations (5.13) can also be obtained by studying the energy, (5.6). Using the identity

$$\big|\vec{D}\phi\big|^2 = \big|(D_1 \pm iD_2)\phi\big|^2 \pm B|\phi|^2 \pm \vec{\nabla} \times \vec{J}$$

and assuming that the fields vanish at infinity, the integral of the current-term can be dropped, so that H becomes

$$\int \Big\{ \frac{1}{2}\Big|(D_1 \pm iD_2)\phi\Big|^2 + \Big[\big(\mp \frac{\gamma}{4\mu} + \frac{\Lambda}{8}\big)(1 - |\phi|^2)^2\Big]\Big\} d^2\vec{x} \pm \underbrace{\tfrac{1}{2}\int B\, d^2\vec{x}}_{\pi n}, \tag{5.18}$$

which shows that the energy is positive definite when the square bracket vanishes, i.e., for the chosen potential with the special value (5.14) of λ. In this case, the energy admits a lower "Bogomolny" bound, $H \geq \pi|n|$, with the equality only attained when the SD equations hold.

Eqn. (5.17) is essentially that of Bogomolny in the Landau-Ginzburg theory [21], to which it reduces when $|\alpha| = 1$. The proofs of Weinberg [22], and of Taubes [14], carry over literally to show, for each n, the *existence of a* $2n$*-parameter family of solutions.*

Radial solutions can be studied numerically [53]; they behave roughly as in the Bogomolny case. Write $\phi = f(r)e^{in\theta}$ where (r, θ) are polar coordinates in the plane. Linearizing the Liouville-type eqn. (5.17), we get for the deviation from the vacuum value, $\varphi = 1 - f$,

$$\varphi'' + \frac{1}{r}\varphi' - |\alpha|\varphi = 0, \tag{5.19}$$

which is Bessel's equation of order zero. The solution and its asymptotic behaviour are therefore

$$1 - \varphi(r) \;\sim\; 1 - K_0(mr) \;\sim\; 1 - \frac{C}{\sqrt{r}}e^{-mr}, \quad m = \sqrt{|\alpha|}. \tag{5.20}$$

It is, however, more convenient to study the first-order equations instead of the Liouville-type eqn. (5.17). For the radial Ansatz

$$a_r = 0, \qquad a_\theta = a(r) \tag{5.21}$$

the self-duality equations read indeed

$$f' = \pm\frac{n + a}{r}f, \qquad \frac{a'}{r} = \pm 2(f^2 - 1). \tag{5.22}$$

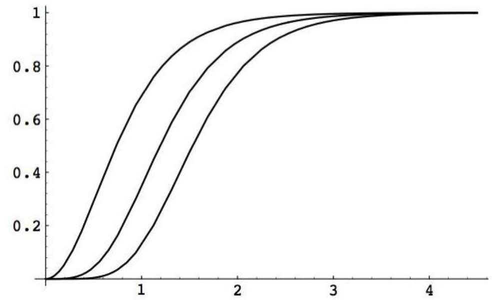

Figure 9: *The scalar field of the radially symmetric Maxwell-Chern-Simons vortices with winding numbers* $n = 1, 2, 3$.

For small r we get

$$f(r) \sim \beta r^{|n|}, \qquad a \sim \mp r^2 \tag{5.23}$$

where β is some real parameter. The large-r behaviour (5.20) of the scalar field is confirmed, and for the magnetic and electric fields we get

$$\begin{aligned} B &= \frac{\alpha}{2}(1-f^2) \sim \alpha \frac{D}{\sqrt{r}} e^{-mr}, && (5.24) \\ \vec{E} &= -\tfrac{1}{4\mu}\big(\mp 1 + \alpha\big)\vec{\nabla} f^2 \sim \frac{G}{\sqrt{r}} e^{-mr}. && (5.25) \end{aligned}$$

Let us mention that the symmetries of the Manton model can be studied along the lines indicated above. The clue is to observe that putting

$$B^{ext} \equiv \frac{\gamma}{2\mu}, \qquad E^{ext}_k = -\frac{\epsilon_{kl} {J^T}_l}{2\mu}, \tag{5.26}$$

transforms the equations of motion (5.2-5.3-5.4) into

$$\begin{aligned} i\gamma D_t \phi &= -\tfrac{1}{2}\vec{D}^2\phi - \tfrac{\lambda}{4}\big(1-|\phi|^2\big)\phi, \\ \epsilon_{ij}\partial_j \widetilde{B} &= J_i + 2\mu\,\epsilon_{ij}\,\widetilde{E}_j, \\ 2\mu\widetilde{B} &= -\gamma|\phi|^2, \end{aligned} \tag{5.27}$$

where

$$\widetilde{B} = B - B^{ext} \qquad \widetilde{E}_i = E_i - E^{ext}_i, \tag{5.28}$$

$$D_\alpha = \partial_\alpha - i a_\alpha, \qquad a_\alpha = \tilde{A}_\alpha + A^{ext}_\alpha. \tag{5.29}$$

These equations describe a non-relativistic scalar field with Maxwell-Chern-Simons dynamics with a symmetry-breaking quartic potential, put into a constant external electromagnetic field. For details and for a discussion of the Manton model to other similar ones [53, 54], the reader is referred to [57].

5.2 Relativistic models and their non-relativistic limit

In relativistic Maxwell-Chern-Simons theory self-dual solutions only arise when an auxiliary neutral field N is added [52]. Here we present a model of this type, which (i) is relativistic; (ii) can be made self-dual; (iii) its non-relativistic limit is the Manton model presented in this paper.

Let us consider in fact (1+2)-dimensional Minkowski space with the metric $(c^2/\gamma, -1, -1)$ where $\gamma > 0$ is a constant. Let us choose the Lagrangian

$$\text{Ł}_R = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{\mu}{2}\epsilon^{\mu\nu\rho}F_{\mu\nu}a_\rho + \big(D_\mu\psi\big)\big(D^\mu\psi\big)^* + a^\mu {J^T}_\mu + \tfrac{\gamma}{2c^2}\partial_\mu N\partial^\mu N - V. \tag{5.30}$$

Here N is an auxiliary neutral field, which we choose real. We have also included the term $a^\mu {J^T}_\mu$, where the Lorentz vector ${J^T}_\mu$ represents the relativistic generalization of Manton's transport current. We choose ${J^T}_\mu$ to be time-like, $\mathrm{I}^2 \equiv \frac{\gamma}{c^2}{J^T}_\mu {J^T}^\mu > 0$. Our choice for the potential is

$$V = \frac{\beta}{2}\big(|\psi|^2 - 2|\mu|N - \frac{\mathrm{I}}{2m\gamma}\big)^2 + \frac{\gamma}{c^2}\big(N + mc^2\big)^2|\psi|^2 - (N + mc^2)\mathrm{I}, \tag{5.31}$$

where $\beta > 0$. Although the potential is *not* positive definite, this will cause no problem when the Gauss law is taken into account, as it will be explained later.

The Lagrangian (5.31) is Lorentz-invariant. The associated equations of motion read

$$\begin{array}{ll}
D_\mu D^\mu\psi + \dfrac{\partial V}{\partial\psi^*} = 0, & \text{non-linear Klein-Gordon eqn.} \\
\frac{\gamma}{c^2}\partial_0 F_{0i} + \epsilon_{ij}\partial_j F_{12} + 2\mu\epsilon_{ij}F_{0j} - J_i + {J^T}_i = 0, & \text{Ampère's law} \\
\frac{\gamma}{c^2}\partial_i F_{0i} + 2\mu F_{12} = \frac{\gamma}{c^2}\big(J_0 - {J^T}_0\big), & \text{Gauss' law} \\
\frac{\gamma}{2c^2}\partial_\mu\partial^\mu N + \dfrac{\partial V}{\partial N} = 0 & \text{auxiliary eqn. for N.}
\end{array} \tag{5.32}$$

In a Lorentz frame where the spatial components of the transport current vanishes, $J^T_\mu = (-\frac{c^2}{\gamma}\mathrm{I}, 0)$. Then, using the Gauss law, we find for the energy

$$H_R = \int d^2\vec{x}\,\Big\{\tfrac{\gamma}{2c^2}\,\vec{E}^2 + \tfrac{1}{2}B^2 + \tfrac{\gamma}{c^2}\,\big|D_0\psi\big|^2 + \big|\vec{D}\psi\big|^2 + \tfrac{\gamma^2}{2c^4}\,\big(\partial_0 N\big)^2 + \tfrac{\gamma}{2c^2}\,\big(\vec{\nabla}N\big)^2 + V\Big\}, \tag{5.33}$$

where we used the obvious notations $E_i = F_{0i}$, $B = F_{12}$ and we have assumed that the surface terms,

$$\tfrac{\gamma}{c^2}\,\vec{\nabla}\cdot\big(a_0\vec{E}\big) + \mu\vec{\nabla}\times\big(a_0\vec{a}\big), \tag{5.34}$$

fall off sufficiently rapidly at infinity. To get finite energy, we require that the energy density go to zero at infinity. Note that $|D_0\psi|^2$ does *not* go to zero at infinity, because

$$J_0 = (-i)\big(D_0\psi\psi^* - \psi(D_0\psi)^*\big)$$

has to go to $J_0^T \neq 0$ at spatial infinity. This term combines rather with the last two terms in the potential. At spatial infinity, the energy density becomes the sum of positive terms. Requiring that all these terms go to zero allows us to conclude that finite energy requires

$$|\vec{E}| \to 0, \qquad B \to 0, \qquad |\psi|^2 \to \frac{\mathrm{I}}{2m\gamma}, \qquad N \to 0. \tag{5.35}$$

Using the Bogomolny trick and the Gauss' law as written in Eqn. (5.32), the term linear in N in the potential gets absorbed. Then the energy is re-written, for the particular value $\beta = 1$,

as

$$
\begin{aligned}
H_R = \int \quad & \Big\{ \tfrac{\gamma}{2c^2} \big[\vec{E} + \vec{\nabla} N\big]^2 + \tfrac{1}{2}\big[B + \epsilon(|\psi|^2 - 2|\mu|N - \tfrac{\mathrm{I}}{2m\gamma})\big]^2 \\
& + \tfrac{\gamma}{c^2}\big|D_0\psi + i(N + mc^2)\psi\big|^2 + \big|(D_1 + i\epsilon D_2)\psi\big|^2 + \tfrac{\gamma^2}{2c^4}\big[\partial_0 N\big]^2 \Big\}\, d^2x \\
& -\epsilon\big(2|\mu|mc^2 - \tfrac{\mathrm{I}}{2m\gamma}\big) \underbrace{\int B\, d^2x}_{\text{flux}},
\end{aligned}
\tag{5.36}
$$

where ϵ is the sign of μ. The last term is topologic, labelled by the winding number, n, of ψ. Due to the presence of c^2, it seems to be reasonable to assume that the coefficient in front of the magnetic flux is positive. Then, chosing $n < 0$ for $\epsilon \equiv \mathrm{sign}(\mu) > 0$ and $n > 0$ for $\epsilon \equiv \mathrm{sign}(\mu) < 0$ respectively, the energy admits hence the "Bogomolny" bound

$$
H_R \geq \big(2|\mu|mc^2 - \frac{\mathrm{I}}{2m\gamma}\big)\, 2\pi|n|. \tag{5.37}
$$

The absolute minimum is attained by those configurations which solve the "Bogomolny" equations

$$
\begin{aligned}
& \partial_0 N = 0, \\
& \vec{\nabla} N + \vec{E} = 0, \\
& D_0\psi + i(N + mc^2)\psi = 0, \\
& \big(D_1 + i\epsilon D_2\big)\psi = 0, \\
& B = \epsilon\big(\tfrac{\mathrm{I}}{2m\gamma} - |\psi|^2 + 2|\mu|N\big).
\end{aligned}
\tag{5.38}
$$

It can also be checked directly that the solutions of these equations solve the second-order field equations (5.32), when the gauge fields are static and the matter field is of the form

$$
\psi = e^{-imc^2 t} \times (\text{static}),
$$

Eqns (5.38) equations are similar to those of by Lee et al., and could be studied numerically as in Ref. [52]. Note that, just like in the case studied by Donatis and Iengo [54], the solutions are *chiral* in that the winding number and the sign of μ are correlated.

Let us stress that for getting a non-zero electrical field, the presence of a non-vanishing auxiliary field N is essential. For $N = 0$ we get rather a self-dual extension of the model of Paul and Khare in Ref. [20], whose vortex solutions are purely magnetic.

Now we show that the non-relativistic limit of our relativistic model presented above is precisely the Manton model. To see this, let us put

$$
\psi = \frac{1}{\sqrt{2m}} e^{-imc^2 t}\, \phi. \tag{5.39}
$$

The transport current is the long-distance limit of the supercurrent, ${J^T}_\mu = \lim_{r\to\infty} J_\mu$. But $\lim_{c\to\infty} J_0/c^2 = -|\phi|^2$, so we have

$$
\lim_{c\to\infty} {J^T}_0/c^2 = -\lim_{r\to\infty} |\phi|^2 = -\lim_{c\to\infty} \frac{\mathrm{I}}{\gamma} \equiv -\alpha. \tag{5.40}
$$

Then the standard procedure yields, after dropping the term $mc^2\mathrm{I}$, the non-relativistic expression

$$\begin{aligned}\mathcal{L}_{NR} = \quad & -\tfrac{1}{2}B^2 + \gamma\tfrac{i}{2}\big(\phi^* D_t\phi - \phi(D_t\phi)^*\big) - \tfrac{1}{2m}\big|\vec{D}\phi\big|^2 \\ & +\mu\big(Ba_t + E_2a_1 - E_1a_2\big) - \gamma a_t - \vec{a}\cdot\vec{J}^T \\ & -\Big\{\tfrac{\beta}{8m}\big(\alpha - |\phi|^2 + 4m|\mu|N\big)^2 - \gamma\big(\alpha - |\phi|^2\big)N\Big\}.\end{aligned} \tag{5.41}$$

Note that there is no kinetic term left for the auxiliary field N. It can therefore be eliminated altogether by using its equation of motion,

$$4\mu^2\beta N = \big(\gamma - \frac{|\mu|\beta}{m}\big)\big(\alpha - |\phi|^2\big). \tag{5.42}$$

Inserting this into the potential, this latter becomes

$$\big(\frac{\gamma}{4|\mu|m} - \frac{\gamma^2}{8\mu^2\beta}\big)\big(\alpha - |\phi|^2\big)^2. \tag{5.43}$$

For $\alpha = 1$ and $m = 1$ in particular, we get precisely the Manton Lagrangian (5.1) with

$$\lambda = \frac{2\gamma}{|\mu|} - \frac{\gamma^2}{\mu^2\beta}. \tag{5.44}$$

The non-relativistic limit of the equations of motion (5.32) is (5.2-5.3-5.11), as it should be.

• In Ampère's law, the first term $(\gamma/c^2)\partial_0F_{0i}$ can be dropped; setting (5.39), the relativistic current becomes the non-relativistic expression $\vec{J} = (1/2i)\big(\phi^*\vec{D}\phi - \phi(\vec{D}\phi)^*\big)$;

• In Gauss' law, the first term $(\gamma/c^2)\partial_iF_{0i}$ can be dropped; the time-component of the currents behave, as already noticed, as

$$\lim_{c\to\infty} J_0/c^2 = -|\phi|^2, \lim_{c\to\infty} {J^T}_0/c^2 = -\alpha = -1.$$

• In the equation for the auxiliary field N the first term $(\gamma/c^2)\partial_\mu\partial^\mu N$ can be dropped and the $c\to\infty$ limit of $\partial V/\partial N = 0$ is (5.42);

• Putting (5.39) into the nonlinear Klein-Gordon equation and using the equation of motions for N, a lengthy but straightforward calculation yields the non-linear Schrödinger equation (5.2), as expected.

Note also that, for the self-dual value $\beta = 1$ (when λ in (5.44) becomes (5.14)), the non-relativistic limit of the (relativistic) self-dual equations (5.38)) fixes a_0 and N as

$$a_0 = N = \big(-\frac{\epsilon}{4\mu} + \frac{\gamma}{4\mu^2}\big)\big(1 - |\phi|^2\big). \tag{5.45}$$

which is consistent with Eq. (5.15). The other equations reduce in turn to our non-relativistic self-dual equations (5.13). At last, subracting mc^2I and taking the limit $c\to\infty$, the relativistic Bogomolny bound (5.37) reduces to the non-relativistic value (5.18).

6 Spinor vortices

6.1 Relativistic spinor vortices

In Ref. [49] Cho et al. obtain, by dimensional reduction from Minkowski space, a $(2+1)$-dimensional system. After some notational changes, their equations read

$$\begin{aligned}&\tfrac{1}{2}\kappa\epsilon^{\alpha\beta\gamma}F_{\beta\gamma} = e\big(\bar{\psi}_+\gamma_+^\alpha\psi_+ + \bar{\psi}_-\gamma_-^\alpha\psi_-\big),\\ &\big(ic\gamma_\pm^\alpha D_\alpha - m\big)\psi_\pm = 0,\end{aligned} \tag{6.1}$$

where the two sets of Dirac matrices are

$$(\gamma^{\alpha}_{\pm}) = (\pm(1/c)\sigma^3, i\sigma^2, -i\sigma^1), \tag{6.2}$$

and the $\psi_\pm$ denote the chiral components, defined as eigenvectors of the chirality operator

$$\Gamma = \begin{pmatrix} -i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix}. \tag{6.3}$$

Observe that, although the Dirac equations are decoupled, the chiral components are still coupled through the Chern-Simons equation. Stationary solutions, representing purely magnetic vortices, are readily found [49]. It is particularly interesting to construct *static* solutions. For $A_0 = 0$ and $\partial_t A_i = 0$, setting

$$\psi_\pm = e^{-imt} \begin{pmatrix} F_\pm \\ G_\pm \end{pmatrix}, \tag{6.4}$$

the *relativistic* system (6.1) becomes, for $c = 1$,

$$\begin{aligned} &\kappa\epsilon^{ij}\partial_i A_j = -e\big(|F_+|^2 + |G_-|^2\big), \\ &\big(D_1 + iD_2\big)F_\pm = 0, \qquad \big(D_1 - iD_2\big)G_\pm = 0. \end{aligned} \tag{6.5}$$

Now, for $F_\pm = 0$ or $G_\pm = 0$, these equations are *identical* to those which describe the non-relativistic, self-dual vortices of Jackiw and Pi [28, 29].

6.2 Non-relativistic spinor vortices

Non-relativistic spinor vortices can also be constructed along the same lines [50]. Following Lévy-Leblond [51], a non-relativistic spin $\frac{1}{2}$ field $\psi = \begin{pmatrix} \Phi \\ \chi \end{pmatrix}$ where Φ and χ are two-component 'Pauli' spinors, is described by the $2+1$ dimensional equations

$$\left\{ \begin{array}{lclcl} (\vec{\sigma}\cdot\vec{D})\,\Phi & + & 2m\,\chi & = & 0, \\ D_t\,\Phi & + & i(\vec{\sigma}\cdot\vec{D})\,\chi & = & 0. \end{array} \right. \tag{6.6}$$

These spinors are coupled to the Chern-Simons gauge field through the mass (or particle) density, $\varrho = |\Phi|^2$, as well as through the spatial components of the current,

$$\vec{J} = i\big(\Phi^\dagger\vec{\sigma}\,\chi - \chi^\dagger\vec{\sigma}\,\Phi\big), \tag{6.7}$$

according to the Chern-Simons equations (3.9). The chirality operator is still given by Eqn. (6.3) and is still conserved. Observe that Φ and χ in Eqn. (6.6) are *not* the chiral components of ψ; these latter are defined by $\frac{1}{2}(1 \pm i\Gamma)\psi_\pm = \pm\psi_\pm$.

It is easy to see that Eqn. (6.6) splits into two uncoupled systems for ψ_+ and ψ_-. Each of the chiral components separately describe (in general different) physical phenomena in $2+1$ dimensions. For the ease of presentation, we keep, nevertheless, all four components of ψ.

Now the current can be written in the form:

$$\vec{J} = \frac{1}{2im}\Big(\Phi^\dagger\vec{D}\Phi - (\vec{D}\Phi)^\dagger\Phi\Big) + \vec{\nabla}\times\Big(\frac{1}{2m}\,\Phi^\dagger\sigma_3\Phi\Big). \tag{6.8}$$

Using the identity

$$(\vec{D}\cdot\vec{\sigma})^2 = \vec{D}^2 + eB\sigma_3,$$

we find that the component-spinors satisfy

$$\left\{\begin{array}{lcl} iD_t\Phi &=& -\dfrac{1}{2m}\Big[\vec{D}^2+eB\sigma_3\Big]\Phi, \\ iD_t\chi &=& -\dfrac{1}{2m}\Big[\vec{D}^2+eB\sigma_3\Big]\chi-\dfrac{e}{2m}\left(\vec{\sigma}\cdot\vec{E}\right)\Phi. \end{array}\right. \tag{6.9}$$

Thus, Φ solves a 'Pauli equation', while χ couples through the term, $\vec{\sigma}\cdot\vec{E}$. Expressing $\vec{E}$ and B through the Chern-Simons equations (4.4-4.5) and inserting into our equations, we get finally

$$\left\{\begin{array}{l} iD_t\Phi = \quad \Big[-\dfrac{1}{2m}\vec{D}^2+\dfrac{e^2}{2m\kappa}\,|\Phi|^2\,\sigma_3\Big]\Phi, \\ iD_t\chi = \quad \Big[-\dfrac{1}{2m}\vec{D}^2+\dfrac{e^2}{2m\kappa}\,|\Phi|^2\,\sigma_3\Big]\chi-\dfrac{e^2}{2m\kappa}\big(\vec{\sigma}\times\vec{J}\big)\Phi. \end{array}\right. \tag{6.10}$$

If the chirality of ψ is restricted to $+1$ (or -1), this system describes non-relativistic spin $+\frac{1}{2}$ ($-\frac{1}{2}$) fields interacting with a Chern-Simons gauge field. Leaving the chirality of ψ unspecified, it describes *two* spinor fields of spin $\pm\frac{1}{2}$, interacting with each other and the Chern-Simons gauge field.

Since the lower component is simply $\chi=-(1/2m)(\vec{\sigma}\cdot\vec{D})\Phi$, it is enough to solve the Φ-equation. For

$$\Phi_+=\begin{pmatrix}\Psi_+\\0\end{pmatrix}\Phi_-=\begin{pmatrix}0\\\Psi_-\end{pmatrix} \tag{6.11}$$

respectively — which amounts to working with the $\pm$ chirality components — the 'Pauli' equation in (6.10) reduces to

$$iD_t\Psi_\pm=\Big[-\frac{\vec{D}^2}{2m}\pm\lambda\,(\Psi_\pm^\dagger\Psi_\pm)\Big]\Psi_\pm, \qquad \lambda\equiv\frac{e^2}{2m\kappa}, \tag{6.12}$$

which again (4.3), but with non-linearity $\pm\lambda$, *half* of the special value Λ in (4.10), used by Jackiw and Pi. For this reason, our solutions (presented below) will be *purely magnetic*, ($A_t\equiv 0$), unlike in the case studied by Jackiw and Pi.

In detail, let us consider the static system

$$\left\{\begin{array}{l} \Big[-\frac{1}{2m}(\vec{D}^2+eB\sigma_3)-eA_t\Big]\Phi=0, \\ \vec{J}=-\dfrac{\kappa}{e}\vec{\nabla}\times A_t, \\ \kappa B=-e\varrho, \end{array}\right. \tag{6.13}$$

and try the first-order Ansatz

$$\big(D_1\pm iD_2\big)\Phi=0 \tag{6.14}$$

that allows us to replace $\vec{D}^2=D_1^2+D_2^2$ by $\mp eB$, then the first equation in (6.13) can be written as

$$\Big[-\frac{1}{2m}eB(\mp1+\sigma_3)-eA_t\Big]\Phi=0, \tag{6.15}$$

while the current is

$$\vec{J}=\frac{1}{2m}\vec{\nabla}\times\Big[\Phi^\dagger(\mp1+\sigma_3)\Phi\Big]. \tag{6.16}$$

Now, due to the presence of σ_3, both Eqn. (6.16) and the second equation in (6.13) can be solved with a *zero* A_t and $\vec{J}$: by choosing $\Phi\equiv\Phi_+$ ($\Phi\equiv\Phi_-$) for the upper (lower) cases

respectively makes $(\mp 1+\sigma_3)\Phi$ vanish. (It is readily seen from Eqn. (6.15) that any solution has a definite chirality). The remaining task is to solve the first-order conditions

$$(D_1 + iD_2)\Psi_+ = 0, \qquad \text{or} \qquad (D_1 - iD_2)\Psi_- = 0, \tag{6.17}$$

which is done in the same way as before :

$$\vec{A} = \pm\frac{1}{2e}\vec{\nabla}\times\log\varrho + \vec{\nabla}\omega, \qquad \triangle\log\varrho = \pm\frac{2e^2}{\kappa}\varrho. \tag{6.18}$$

A normalizable solution is obtained for Ψ_+ when $\kappa < 0$, and for Ψ_- when $\kappa > 0$. (These correspond to attractive non-linearity in Eqn. (6.12)). The lower components vanish in both cases, as seen from the χ-equation

$$\chi = -\frac{1}{2m}(\vec{\sigma}\cdot\vec{D})\Phi. \tag{6.19}$$

Both solutions only involve *one* of the $2+1$ dimensional spinor fields $\psi_\pm$, depending on the sign of κ.

The physical properties such as symmetries and conserved quantities can be studied by noting that our equations are in fact obtained by variation of the $2+1$-dimensional action given in [50], which can also be used to show that the coupled Lévy-Leblond — Chern-Simons system is, just like its scalar counterpart, Schrödinger symmetric [29].

A conserved energy-momentum tensor can be constructed and used to derive conserved quantities [50]. One finds that the 'particle number' N determines the actual values of all the conserved charges: for the radially symmetric solution, e.g., the magnetic flux, $-eN/\kappa$, and the mass, $\mathcal{M} = mN$, are the same as for the scalar soliton of [29]. The total angular momentum, however, can be shown to be $I = \mp N/2$, *half* of the corresponding value for the scalar soliton. As a consequence of self-duality, our solutions have *vanishing energy*, just like the ones of Ref. [29].

It is worth mentionning that our non-relativistic spinor model here can in fact be derived from the relativistic theory of Cho et al [49]. Putting

$$\psi_+ = e^{-imc^2t}\begin{pmatrix}\Psi_+\\ \widetilde{\chi}_+\end{pmatrix}\psi_- = e^{-imc^2t}\begin{pmatrix}\widetilde{\chi}_-\\ \Psi_-\end{pmatrix}, \tag{6.20}$$

their Eqn. (6.1) become

$$\begin{cases} iD_t\Phi - c\vec{\sigma}\cdot\vec{D}\widetilde{\chi} = 0, \\ iD_t\widetilde{\chi} + c\vec{\sigma}\cdot\vec{D}\Phi + 2mc^2\widetilde{\chi} = 0, \end{cases} \tag{6.21}$$

where $\Phi = \begin{pmatrix}\Psi_+\\ \Psi_-\end{pmatrix}$ and $\widetilde{\chi} = \begin{pmatrix}\widetilde{\chi}_-\\ \widetilde{\chi}_+\end{pmatrix}$. In the non-relativistic limit

$$mc^2\widetilde{\chi} >> iD_t\widetilde{\chi},$$

so that this latter can be dropped from the second equation. Redefining $\widetilde{\chi}$ as $\chi = c\widetilde{\chi}$ yields precisely our Eqn. (6.6). This also explains, why one gets the same (namely the Liouville) equation both in the relativistic and the non-relativistic cases: for static and purely magnetic fields, the terms containing D_t are automatically zero.

It is worth mentionning that the (2+1) dimensional spinor model presented here can also be obtained in the Kaluza-Klein-type framework of Section 4.4. The Lévy-Leblond equations (6.6)

arises, in particular, as lightlike reduction of the *massless Dirac equation* for a 4-component Dirac spinor on on "Bargmann space" M,

$$\not{D}\psi = 0. \tag{6.22}$$

This framework allows one to rederive the Schrödinger symmetry of the spinor system along the same lines as in the scalar case [50].

6.3 Spinor vortices in nonrelativistic Maxwell-Chern-Simons theory

Now we generalize our construction to non-relativistic Maxwell-Chern-Simons theory of Manton's type. Let Φ denote a 2-component Pauli spinor. We posit the following equations of motion.

$$\begin{cases} i\gamma D_t\Phi = -\frac{1}{2}\big[\vec{D}^2 + B\sigma_3\big]\Phi & \text{Pauli eqn.} \\ \epsilon_{ij}\partial_j B = J_i - J_i^T + 2\mu\,\epsilon_{ij}\,E_j & \text{Ampère's eqn.} \\ 2\mu B = \gamma\big(1-|\Phi|^2\big) & \text{Gauss' law} \end{cases} \tag{6.23}$$

where the current is now

$$\vec{J} = \frac{1}{2i}\Big(\Phi^\dagger\vec{D}\Phi - (\vec{D}\Phi)^\dagger\Phi\Big) + \vec{\nabla}\times\Big(\frac{1}{2}\,\Phi^\dagger\sigma_3\Phi\Big). \tag{6.24}$$

The system is plainly non-relativistic, and it admits self-dual vortex solutions, as we show now. The transport current can again be eliminated by a galilean boost. For fields which are static in the frame where $\vec{J}^T = 0$, the equations of motion become

$$\begin{cases} \big[\frac{1}{2}(\vec{D}^2 + B\sigma_3) + \gamma a_t\big]\Phi = 0, \\ \vec{\nabla}\times B = \vec{J} + 2\mu\,\vec{\nabla}\times a_t, \\ 2\,(\frac{\mu}{\gamma})B = 1 - \Phi^\dagger\Phi. \end{cases} \tag{6.25}$$

Now we attempt to solve these equations by the first-order Ansatz

$$\big(D_1 \pm iD_2\big)\Phi = 0. \tag{6.26}$$

Eqn. (6.26) implies that

$$\vec{D}^2 = \mp B\vec{J} = \tfrac{1}{2}\vec{\nabla}\times\Big[\Phi^\dagger(\mp 1 + \sigma_3)\Phi\Big], \tag{6.27}$$

so that the Pauli equation in (6.25) requires

$$\Big[(\mp 1 + \sigma_3)B + 2\gamma a_t\Big]\Phi = 0. \tag{6.28}$$

Let us decompose Φ into chiral components,

$$\Phi = \Phi_+ + \Phi_- \qquad \text{where} \qquad \Phi_+ = \begin{pmatrix} 0 \\ \chi \end{pmatrix}, \qquad \Phi_- = \begin{pmatrix} \varphi \\ 0 \end{pmatrix}. \tag{6.29}$$

Eqn. (6.28) requires that Φ have a definite chirality. One possibility would be $\Phi_+ = 0$ for the upper sign, and $\Phi_- = 0$ for the lower sign. In both cases, a_t would vanish. These choices are, however, seen to be inconsistent with Ampère's law.

Curiously, there is another possibility : one can have

$$a_t = \pm\tfrac{1}{\gamma}\,B, \quad \text{and} \quad \left\{\begin{array}{lll} \Phi_- = 0 & \text{i.e. } \Phi \equiv \Phi_+ & \text{for the upper sign} \\ \\ \Phi_+ = 0 & \text{i.e. } \Phi \equiv \Phi_- & \text{for the lower sign} \end{array}\right. . \tag{6.30}$$

Then $\vec{J} = \mp\vec{\nabla}\times\big|\Phi_\pm\big|^2$, so that Ampère's law requires

$$\vec{\nabla}\times\Big(\big[1\mp\tfrac{2\mu}{\gamma}\big]B \pm \big|\Phi_\pm\big|^2\Big) = 0. \tag{6.31}$$

But now $\big|\Phi_\pm\big|^2 = \big|\Phi\big|^2 = 1-(2\mu/\gamma)B$ by the Gauss law, so that (6.31) holds when

$$\alpha \equiv \pm\frac{\gamma}{\mu} = 4. \tag{6.32}$$

In conclusion, for the particular value (6.32), the second-order field equations can be solved by solving one or the other of the first-order equations in (6.25). These latter conditions fix moreover the gauge potential as

$$\vec{a} = \pm\tfrac{1}{2}\vec{\nabla}\times\log\varrho + \vec{\nabla}\omega, \qquad \varrho \equiv \big|\Phi\big|^2 = \big|\Phi_\pm\big|^2 \tag{6.33}$$

and then the Gauss law yields

$$\triangle\log\varrho = 4(\varrho-1), \tag{6.34}$$

which is again the "Liouville-type" equation (5.17) studied before. Note that the sign, the same for both choices, is automatically positive, as $\alpha = 4$.

The equations of motion (6.23) can be derived from the Lagrangian

$$\begin{aligned}\mathcal{L} = \quad & -\tfrac{1}{2}B^2 + \frac{i\gamma}{2}\big[\Phi^\dagger(D_t\Phi) - (D_t\Phi)^\dagger\Phi\big] - \frac{1}{2}(\vec{D}\Phi)^\dagger(\vec{D}\Phi) \\ & + \frac{B}{2}\Phi^\dagger\sigma_3\Phi + \mu\big(Ba_t + E_2a_1 - E_1a_2\big) - \gamma a_t - \vec{a}\cdot\vec{J}^T.\end{aligned} \tag{6.35}$$

Then, in the frame where $\vec{J}^T = 0$, the energy is

$$H = \frac{1}{2}\int\Big\{B^2 + \big|\vec{D}\Phi\big|^2 - B\,\Phi^\dagger\sigma_3\Phi\Big\}\,d^2\vec{x}. \tag{6.36}$$

Using the identity

$$\big|\vec{D}\Phi\big|^2 = \big|(D_1\pm iD_2)\Phi\big|^2 \pm B\,\Phi^\dagger\Phi \tag{6.37}$$

(valid up to surface terms), the energy is rewritten as

$$H = \tfrac{1}{2}\int\Big\{B^2 + \big|(D_1\pm iD_2)\Phi\big|^2 - B\Big[\Phi^\dagger(\mp1+\sigma_3)\Phi\Big]\Big\}\,d^2\vec{x}.$$

Eliminating B using the Gauss law, we get finally, for purely chiral fields, $\Phi = \Phi_\pm$,

$$H = \tfrac{1}{2}\int\Big\{\big|(D_1\pm iD_2)\Phi_\pm\big|^2 + \frac{\gamma}{4\mu}\big[\mp4+\frac{\gamma}{\mu}\big]\big(1-|\Phi_\pm|^2\big)^2\Big\}\,d^2\vec{x} \pm \int B\,d^2\vec{x}. \tag{6.38}$$

The last integral here yields the topological charge $\pm2\pi n$. The integral is positive definite when $\pm\gamma/\mu \geq 4$, depending on the chosen sign, yielding the Bogomolny bound

$$H \geq 2\pi|n|. \tag{6.39}$$

The Pauli term hence *doubles* the Bogomolny bound with respect to the scalar case. The bound can be saturated when $\pm\gamma/\mu = 4$ and the self-dual equations (6.26) hold.

7 Conclusion and outlook

In this paper, we reviewed some aspects of Abelian Chern-Simons theories. For completeness, we would like to list a number of related issues not covered by us here.

First of all, much of the properties studied here can be generalized to non-Abelian interactions [64] which have, of course, many further interesting aspects. The Jackiw-Pi vortices, for example, can be generalized to $SU(N)$ gauge theory leading to generalizations of the Liouville equation. See, e.g., Refs. [42, 27].

Experimentally, superconducting vortices arise in fact often as lattices in a finite domain. Within the Jackiw-Pi model, this amounts to selecting doubly-periodic solutions of the Liouville equations [65].

The relation to similar models which arise in condensed matter physics could also be developped [66]. Other interesting aspects concern is anomalous coupling [67], as well as various self-duality properties [68, 69].

Returning to the abelian context, we should mention the study on the dynamics of vortices [70, 71, 72].

Let us mention, in conclusion, recent work on vortices in the non-commutative, “Moyal” field theory [73] as well as the recent review [74].

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